A ‘brief’ discussion of torsors

1 Introduction and motivation

1.1 Introduction

This note was written as supplementary material for an ‘independent study’ I was overseeing at Berkeley for two students studying the étale fundamental group.

The goal of the note is to try and motivate cohomology and torsors and develop, in a somewhat detailed manner, the basic results of the theory. Specifically, to define G-torsors and understand their relationship to H^1—the idea being that the two concepts are the obverse and reverse of the same coin. The perspective of H^1 as cocycles (which will be our base definition) lends itself to the computational. The perspective of H^1 as torsors (which will be the non-trivial relationship we’ll have to show) will highlight the more intuitive, easy to geometrically manipulate aspect of (first) cohomology. We will then discuss the theory of ‘twists’ (which certainly requires more prerequisite knowledge) which generalizes both perspectives.

Specifically, we will start with an introduction to sheaves and cohomology in the abstract. Since the notes were written with étale cohomology in mind, the main example/motivation will be that of the étale topology (with the flat topology also making a cameo), although we will discuss most of the results in the abstract context of sites to highlight the true nature of the constructions. We will then discuss torsors first as sheaves, then as spaces. We will then explain the precise relationship between torsors and first cohomology. Finally, we explain the theory of twists which, to me, is where the true motivation/power in the idea of torsors lies.

1.2 Motivation

In this section we’d like to give a motivation for the reader to read this note—an answer to the question ‘why do I care?’ In particular, this is not meant to be an overview or explain the intuition of the structural properties of torsors but, instead, explain why they are a natural object to create when trying to answer the following exceedingly simple question: what does H^1 ‘mean’? If a reader is not familiar with cohomology it’s advised they read this section *after* §2.

Before we jump into it, it’s helpful to first set ourselves up by answering an even simpler question: what does H^0 mean? Namely, let $\mathcal{F}$ be a sheaf on a space X. Suppose further that U is an open in X and $\{U_i\}$ is an open cover U. Then what precisely is $\check{H}^0(\{U_i\},\mathcal{F})$—what is it ‘parameterizing/measuring’? Well, being good students of algebraic geometry an obvious answer pops into our heads—it’s just $\mathcal{F}(U)$! Indeed, Čech cohomology (at least in low degrees) is just like normal cohomology and what H^0 is just, well, global sections. So, $H^0(U,\mathcal{F})$, which is ‘basically’ $\check{H}^0(\{U_i\},\mathcal{F})$, is just $\mathcal{F}(U)$. Of course, this misses the point entirely. Namely, while it’s true, we’re ignoring what *makes* it true and this obfuscates the whole picture and makes generalization harder.

So, let’s reevaluate the statement that $\check{H}^0(\{U_i\},\mathcal{F})$ is $\mathcal{F}(U)$. Well, first, let’s try to be slightly more precise in this claim. It’s not even true that these two things are literally equal (things never really are) but, what is true, is that there is a canonical map

$$\mathcal{F}(U) \to \check{H}^0(\{U_i\},\mathcal{F}) \tag{1}$$

which is an isomorphism. But, again even though the map (1) is a canonical isomorphism, it’s somewhat harmful to our current goal to identify the two objects—they are not the same. So, if we can’t think of $\check{H}^0(\{U_i\},\mathcal{F})$ as $\mathcal{F}(U)$, how should we think of it? The answer is even simpler than one might originally imagine: $\check{H}^0(\{U_i\},\mathcal{F})$ is *gluing data* for $\mathcal{F}$ with respect to the cover $\{U_i\}$ of U. Namely, it says that we have a collection of elements $s_i \in \mathcal{F}(U_i)$ which are candidates to be glued together to an element of $\mathcal{F}(U)$ since the obvious impediment to such a gluing (that the objects are equal on the intersections) is a non-issue. The point then, the reason that (1) is a canonical isomorphism, is not some inherent fact of the universe, but due precisely to the property that $\mathcal{F}$ is a sheaf.

This then gives an avenue by which we might try to understand objects like $\check{H}^1(\{U_i\}, \mathcal{F})$ when $\mathcal{F}$ is a sheaf of (possibly non-abelian) groups—maybe this first cohomology group is *also* gluing data of some sort. But, gluing data of *what*? Again, dealing with the world of sheaves (of sets) has spoiled us, and so blinds us to what the natural setting H^1 is gluing data in. We will have to leave the comfortable world of sheaves of sets and, instead, enter a world of sheaves of *categories*. Namely, the answer to 'what is $\check{H}^1(\{U_i\}, \mathcal{F})$ is gluing data' is that it's gluing data for objects of a *category*—it's '1-gluing data' whereas $\check{H}^0(\{U_i\}, \mathcal{F})$ is '0-gluing data'. Somewhat confusingly though, it is not $\mathcal{F}$ that will be the sheaf of categories (it's a sheaf of groups!) instead there will be another ambient 'sheaf of categories' $\mathscr{S}$ that $\mathcal{F}$ will depend on (and conversely).

OK, this is all starting to sound a bit outlandish, so let's be specific as to what we mean. Namely, let's suspend disbelief for a second and pretend we know what a 'sheaf of categories' $\mathscr{S}$ on X means. As a key example to keep in mind, perhaps $\mathscr{S}$ is the 'sheaf' that assigns to an open $U \subseteq X$ the category $\mathscr{S}(U)$ of line bundles on U. So, again we're going to be interested in gluing data for $\mathscr{S}$ or, more precisely, gluing data for the *isomorphism classes* of $\mathscr{S}$. Now, if $\mathscr{S}$ were actually a sheaf of sets (note that every set is a category with the only arrows being the identity arrows) the notion of 'isomorphism' is a silly one: if two objects $x, y \in \mathscr{S}(U)$ are 'isomorphic' this just means that they're equal! But, once we move to complicated sheaves of categories (e.g. the sheaf of line bundles as indicated above) then notions of isomorphism become more complicated. Specifically, being equal is *not* the same thing as being isomorphic!

The real complication this brings to the table is the following. If $\mathscr{S}$ is a sheaf of sets, then $x, y \in \mathscr{S}(U)$ being 'locally isomorphic' (i.e. there exists a cover $\{U_i\}$ of U such that $x\mid_{U_i} \cong y\mid_{U_i}$) implies, since isomorphism is the same as equality, that x and y are actually isomorphic. If $\mathscr{S}$ is now just a sheaf of categories with non-trivial isomorphisms, then this no longer needs to hold. As an extreme example, note that if $\mathscr{S}$ is the sheaf of line bundles, then any two objects of $\mathscr{S}(U)$ are locally isomorphic—everything is definitionally locally isomorphic to $\mathcal{O}_X$! So, now that we are dealing with this more complicated situation, the first thing we might imagine doing is creating some sort of qualitification that measures how badly the statement 'locally isomorphic implies globally isomorphic' fails. Namely, if we fix some object $x_0 \in \mathscr{S}(U)$ we might try and figure out how to calculate the set of isomorphism classes of objects $x \in \mathscr{S}(U)$ that are locally isomorphic to x_0.

OK, so where does cohomology come into the picture? And, how is this gluing data? Well, let's pretend for a second that we're interested in the cohomology of a special type of sheaf. Namely, let's suppose that we have our sheaf of categories $\mathscr{S}$ and our fixed object $x_0 \in \mathscr{S}(U)$. Let us then define a sheaf on the open subsets of U, denoted $\mathrm{Aut}(x_0)$, defined by $\mathrm{Aut}(x_0)(V) := \mathrm{Aut}(x_0\mid_V)$ where $x_0\mid_V$ is the image of x_0 under the map (of categories) $\mathscr{S}(U) \to \mathscr{S}(V)$ for an open $V \subseteq U$. I claim then that $H^1(U, \mathrm{Aut}(x_0))$ is *precisely* measuring the failure of the claim 'locally isomorphic to x_0 implies globally isomorphic to x_0' because it precisely classifies the isomorphism classes of objects of $\mathscr{S}(U)$ that are locally isomorphic to x_0.

OK, how is this so? We saw intuitively above that what stops the statement 'locally isomorphic implies globally isomorphic' from holding for $\mathscr{S}$ was that whereas sheaves of sets had an unambiguous notion of isomorphism, general sheaves of categories don't. Namely, if I tell you that $\{U_i\}$ is a cover of U such that $x\mid_{U_i} \cong x_0\mid_{U_i}$ for $x \in \mathscr{S}(U)$ then a lot has been left unsaid. Namely, there are *lots* of isomorphisms $x\mid_{U_i} \xrightarrow{\approx} x_0\mid_{U_i}$ and its precisely this ambiguity that stops us gluing these local isomorphisms to a global one—if there was only one isomorphism then (exercise!) one could glue these local isomorphisms to a global one. But, where $\mathrm{Aut}(x_0)$ comes into the picture is the realization that while there are many isomorphisms $x\mid_{U_i} \xrightarrow{\approx} x_0\mid_{U_i}$ all of them differ by an element of $\mathrm{Aut}(x_0)(U_i) = \mathrm{Aut}(x_0\mid_{U_i})$. Thus, it seems reasonable to expect that how one obtains the object x is by some 'gluing' involving these various elements of $\mathrm{Aut}(x_0)(U_i)$—this makes it seem plausible that these objects x are classified by a cohomology group with coefficients in $\mathrm{Aut}(x_0)$. But, why is it *first* cohomology?

Well, since H^1 is the same thing as Čech cohomology (i.e. $\check{H}^1$) and Čech cohomology is the colimit of the cohomologies over all covers (i.e. $\check{H}^1(\{U_i\}, -)$) it really suffices for us to understand what $\check{H}^1(\{U_i\}, \mathrm{Aut}(x_0))$ is doing for $\{U_i\}$ a cover of U. So, let us explain how an object $x \in \mathscr{S}(U)$ such that $x\mid_{U_i} \cong x_0\mid_{U_i}$ gives rise to an element of $\check{H}^1(\{U_i\}, \mathrm{Aut}(x_0))$. Indeed, since $x\mid_{U_i} \cong x_0\mid_{U_i}$ there exists an isomorphism $\varphi_i : x\mid_{U_i} \to x_0\mid_{U_i}$. Now, if these φ_i's agreed on intersections then we could glue them together to an isomorphism $x \xrightarrow{\approx} x_0$, so the next sensible thing to think about is how these φ_i's interact on overlaps. In particular, note that on $U_i \cap U_j$ we obtain two isomorphisms $\varphi_i\mid_{U_{ij}} : x\mid_{U_{ij}} \to x_0\mid_{U_{ij}}$ and $\varphi_j : x\mid_{U_{ij}} \to x_0\mid_{U_{ij}}$. And, as mentioned above, these differ by an automorphism of $x_0\mid_{U_{ij}}$, namely $(\varphi_i\mid_{U_{ij}}) \circ (\varphi_j\mid_{U_{ij}})^{-1}$. Thus, by thinking about how these isomorphisms compare on overlaps we've obtained an element $(s_{ij}) = ((\varphi_i\mid_{U_{ij}}) \circ (\varphi_j\mid_{U_{ij}})^{-1})$of $\displaystyle\prod_{ij} \mathrm{Aut}(x_0)(U_{ij})$. Note, moreover, that these elements (s_{ij}) satisfy the property that

$$(s_{ij} \mid_{U_{ijk}}) \circ (s_{jk} \mid_{U_{ijk}}) = s_{ik} \mid_{U_{ijk}} \tag{2}$$

Thus, if

$$\partial_1 : \prod_{ij} \mathrm{Aut}(x_0)(U_{ij}) \to \prod_{ijk} \mathrm{Aut}(x_0)(U_{ijk}) \tag{3}$$

denotes the usual differential (at the 1-term) of the Čech complex, then we see that how the isomorphisms φ_i interact on intersections gives rise to an element $(s_{ij}) \in Z^1(\{U_i\}, \mathrm{Aut}(x_0)) := \ker \partial_1$. Or, in different parlance, checking how the isomorphisms φ_i interact on overlaps gives rise to a 1-cocycle for $\mathrm{Aut}(x_0)$ on the covering $\{U_i\}$.

Now, suppose that we had chosen different isomorphisms $\psi_i : x \mid_{U_i} \to x_0 \mid_{U_i}$ because, after all, we're interested in objects x locally isomorphic to x_0, not *how* they're locally isomorphic. Note then that by considering the compositions $s_i := \varphi_i \circ \psi_i^{-1}$ we obtain an element of $\prod_i \mathrm{Aut}(x_0)(U_i)$. Moreover, one can check that the two 1-cocycles (s_{ij}^{φ}) and (s_{ij}^{ψ}) differ from one another by the image of (s_i) under the map

$$\partial_0 : \prod_{i} \mathrm{Aut}(x_0)(U_i) \to \prod_{ij} \mathrm{Aut}(x_0)(U_{ij}) \tag{4}$$

Thus, if $B^1(\{U_i\}, \mathrm{Aut}(x_0)) := \mathrm{im}\, \partial^0$, the 1-coboundaries, then we deduce that an x such that $x \mid_{U_i} \cong x_0 \mid_{U_i}$ unambiguously defines an element of $Z^1(\{U_i\}, \mathrm{Aut}(x_0))/B^1(\{U_i\}, \mathrm{Aut}(x_0))$ or, in other words, defines an element of $\check{H}^1(\{U_i\}, \mathrm{Aut}(x_0))$.

Thus, to summarize we see that isomorphism classes of objects of $\mathscr{S}(U)$ which become ismorphic to x_0 on the cover $\{U_i\}$ give rise to elements of $\check{H}^1(\{U_i\}, \mathrm{Aut}(x_0))$ and, even though we haven't show this (it's not that hard of an exercise), the converse is also true. So, we see that $\check{H}^1(U, \mathrm{Aut}(x_0))$ (which is obtained by letting the cover $\{U_i\}$ vary) classifies isomorphisms classes of objects x of $\mathscr{S}(U)$ locally isomorphic to x_0. Moreover, one can see that under this light $\check{H}^1(U, \mathrm{Aut}(x_0))$ is, in fact, measuring discrepancies in gluing data. Namely, one sees that the element $(s_{ij}) \in \check{H}^1(\{U_i\}, \mathrm{Aut}(x_0))$ associated to x is exactly giving the inability to glue the isomorphisms $x \mid_{U_i} \cong x_0 \mid_{U_i}$ together to get an isomorphism $x \cong X_0$. It's all just gluing data except now it's gluing data for *morphisms* not *objects* (i.e. it's 1-gluing data)!

Now, while this is really nice, it seems to unify $\check{H}^0$ and $\check{H}^1$ and suggest further generlization (maybe $\check{H}^i$ is 'higher order' gluing data) it has one serious flaw. Namely, this only gives a nice interpration to the cohomology on U of a group sheaf $\mathcal{G}$ if it's of the form $\mathrm{Aut}(x_0)$ for for some object $x_0 \in \mathscr{S}(U)$ for $\mathscr{S}$ a sheaf of categories. What do we do if $\mathcal{G}$ is a group sheaf *not* of this form? Well, perhaps a better a question is 'can we find for any $\mathcal{G}$ a canonical $\mathscr{S}$ and $x_0 \in \mathscr{S}(U)$ such that $\mathcal{G} = \mathrm{Aut}(x_0)$?'. The answer is, pleasingly, *yes*. Namely, the category of sheaves $\mathscr{S}$ will be the category of '$\mathcal{G}$-torsors' and the object $x_0 \in \mathscr{S}(U)$ will be the 'trivial $\mathcal{G}$-torsor'. What makes this solution so incredibly satisfying is that torsors (whatever they are!) are not just random objects, they are beautifully intricate geometric structures interesting in their own right. But, because they allow us to realize every $\mathcal{G}$ as $\mathrm{Aut}(x_0)$ they, combined with the discussion from the last few paragraphs, allows us to answer our original question: what does $\check{H}^1(U, \mathcal{G})$ classify? It classifies *torsors*.

Remark 1.1: Note that for a group sheaf $\mathcal{G}$ one can, *a priori*, find many pairs $(x_0, \mathscr{S})$ realizing $\mathcal{G}$ as $\mathrm{Aut}(x_0)$. This perspective is extremely powerful, and will allow us to use cohomology to compute some really interesting things. So, while 'torsors' are the canonical way of making a pair $(x_0, \mathscr{S})$ one shouldn't think it's the only one. We'll see this more in §5. ♦

2 Sheaves and cohomology

2.1 Introduction

Like it or not, cohomology is everywhere. It is the language of measuring obstructions in mathematics. What I mean by this is that if an object X doesn't satisfy property P then a next natural question is, well, how *badly* does it fail to satisfy property P? Is there some sort of qualitative/quantitative measure of this failure? Most of the time the answer is yes, and most of the time the answer is encapsulated in a cohomology group.

Since it will be important here, let me recall one of the most general settings that cohomology takes place in. The vast majority of the cohomology theories that you know arise in this context. What context? The context of sheaves (as opposed to something like singular or celluar cohomology). But, to fully take advantage of this fruitful line of thought, we will need to indulge in a bit of psychedelia and expand our minds—we need a much more general notion of sheaves than we're used to. Specifically, we'll need to think about what are called *sites*.

The idea of sites is somewhat obvious. Namely, Grothendieck and his collaborators realized that while we usually think about sheaves on a topological space X, one doesn't *really* need a topology. Namely, if one thinks about it, a sheaf $\mathcal{F}$ is just the rigorization of the idea of assigning sets $\mathcal{F}(X)$ for every object X and such that any time one has a cover $\{U_i \to X\}$ one has that the objects of $\mathcal{F}(X)$ are precisely the objects of $\prod_i \mathcal{F}(U_i)$ that agree on intersections. In other words, to talk about sheaves we really only need to be able to talk about coverings.

The reason that such flights of fancy are useful in practice is that often times in algebraic geometry we want to think about a scheme as being endowed with a 'better topology' than the usual old Zariski topology. The reason is somewhat clear. Namely, if one hands you $\mathbb{A}^1_\mathbb{C}$ you already have an idea of what the open subsets of $\mathbb{A}^1_\mathbb{C}$ 'look like'—we still draw them as balls or intervals (depending if we're drawing $\mathbb{A}^1_\mathbb{C}$ as a line or the plane)—even though the actual topology of $\mathbb{A}^1_\mathbb{C}$ has only HUGE open sets—the complements of finitely many points. As we are about to see, one can 'access' this (or at least a) 'better' topology for schemes (in an algebraic way) but only via the notion of sites. This itself should justify why such ideas are worth considering.

2.2 Motivation for étale topology

So, while we're likely doomed to be able to talk about a finer topology than the Zariski topology in an algebro-geometric way if we stick to literal closed/open subsets of $\mathbb{A}^1_\mathbb{C}$ if we are able to consider special types of maps $X \to \mathbb{A}^1_\mathbb{C}$ as being 'generalized opens' we can get a theory that acts similarly to how we'd hope a 'correct topology' (one like the one we draw) acts. To understand what we mean by 'acts similarly' let us give some examples of sheaf theory in the classical topology of $\mathbb{A}^1_\mathbb{C}$ that will help guide us.

Namely, consider the function $\mathbb{C}^\times \to \mathbb{C}^\times$ given by $z \mapsto z^2$. Then, locally for the usual (complex) topology on $\mathbb{C}$, this function has an inverse. For example, if we restrict to the right half-plane $\{z : \mathrm{Re}(z) > 0\}$ then one can actually write down a branch of the logarithm $\log(z)$ in which case $\exp(\frac{1}{2}\log(z))$ is an inverse to $z \mapsto z^2$. Why do we care about this? Well, because it (or natural extensions of the techniques used to show it) shows that while the map $\mathcal{O}^\times_\mathbb{C}(U) \to \mathcal{O}^\times_\mathbb{C}(U)$ (here $\mathcal{O}_\mathbb{C}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}^\times_C$ is its sheaf of units) is not surjective for all U (for example putting $U = \mathbb{C}^\times$ won't work since $z \mapsto z^2$ does <u>not</u> globally have an inverse) it is surjective on small opens. In other words, the map of sheaves $\mathcal{O}^\times_\mathbb{C} \to \mathcal{O}^\times_\mathbb{C}$ given by $f \mapsto f^2$ (more rigorously the map which on every U sends a non-vanishing function f to its square) <u>is</u> surjective!

This line of thinking turns out to be a great boon. Because the topology on $\mathbb{C}$ is fine enough to allow for a reasonable interaction between highly different sheaves—we are able to bridge the gap between many ostensibly unrelated fields of math on $\mathbb{C}$ using sheaves.

As an example of this, like $z \mapsto z^2$ locally has an inverse, so does $z \mapsto \exp(z)$. So the map $\exp : \mathcal{O}_\mathbb{C} \to \mathcal{O}^\times_\mathbb{C}$ (of course $\exp(f(z))$ is always a unit—it's nowhere vanishing!) is surjective. Moreover, it's kernel is something well-known. Namely, you probably remember from complex analysis that $\exp(2\pi i) = 0$ and, more generally, $\exp(2\pi i n) = 0$ for any $n \in \mathbb{Z}$ (and these are its only zeros). For this reason, the kernel of $\exp : \mathcal{O}_\mathbb{C} \to \mathcal{O}^\times_\mathbb{C}$ is $\underline{2\pi i\mathbb{Z}}$—the constant sheaf of the group $2\pi i\mathbb{Z}$. Thus, we see that we have a short exact sequence of sheaves

$$0 \to \underline{2\pi i\mathbb{Z}} \to \mathcal{O}_\mathbb{C} \to \mathcal{O}_{\mathbb{C}^\times} \to 0 \tag{5}$$

which turns out to relate two separate properties of $\mathbb{C}$.

Namely, the sheaf $\underline{2\pi i\mathbb{Z}}$ is inherently topological in nature. Remember that it's value on U is $(2\pi i\mathbb{Z})^{\pi_0(U)}$ (where $\pi_0(U)$ is the number of connected components). The sheaves $\mathcal{O}_\mathbb{C}$ and $\mathcal{O}^\times_\mathbb{C}$ are inherently analytic—they have to do with holomorphic functions. Thus, the power of sheaves has allowed us to relate topology and analysis in a highly non-trivial way which turns out to be hugely influential on the subject—it actually implies a highly non-trivial interaction between the singular cohomology (a topological cohomology) of X, its coherent cohomology of a structure sheaf (how hard it is to glue holomorphic functions together), and its (holomorphic) line bundles—more on this later.

So, given the success of this we'd like to replicate it in the algebraic world. We quickly, and with great force, hit a brick wall though. Namely, if we consider something like $\mathbb{A}^1_\mathbb{C}$ the topology is <u>so bad</u> that the map of sheaves $\mathcal{O}^\times_{\mathbb{A}^1_\mathbb{C}} \to \mathcal{O}^\times_{\mathbb{A}^1_\mathbb{C}}$ sending f to f^2 (on an open U) is <u>not surjective</u>.

Exercise 2.1: *Prove this claim (NB: the solution is stated immediately after this exercise).*

In fact, if we look at the map $\mathcal{O}^\times_{\mathbb{A}^1_\mathbb{C}}(D(T)) \to \mathcal{O}^\times_{\mathbb{A}^1_\mathbb{C}}(D(T))$ and look at the element T in the codomain (thinking of $\mathbb{A}^1_\mathbb{C} = \operatorname{Spec}(\mathbb{C}[T])$) there is no Zariski open cover $\{U_i\}$ of $D(T)$ such that $T\mid_{U_i}$ is a square for all i. We see directly that the lack of extremely simple maps of sheaves (i.e. the squaring map!) being surjective is due to a deficit of open subsets of $\mathbb{A}^1_\mathbb{C}$—we need a richer topology.

That said, we don't need to think that hard. Namely, there is a very natural way to obtain a square root of T! How? Well, consider the map $f : \operatorname{Spec}(A) \to D(T)$ where $A = \mathbb{C}[T, T^{-1}, Y]/(Y^2 - T)$ and the map $\operatorname{Spec}(A) \to D(T)$ corresponds to the inclusion $\mathbb{C}[T, T^{-1}] \hookrightarrow \mathbb{C}[T, T^{-1}, Y]/(Y^2 - T)$. Note then that if we consider the 'restriction' $T\mid_{\operatorname{Spec}(A)}$ (more rigorously the image of T under the map on global sections of sheaves $\mathcal{O}_{D(T)} \to f_*\mathcal{O}_{\operatorname{Spec}(A)}$) it now does have a square-root—Y! Thus, if we were able to think of $\operatorname{Spec}(A) \to D(T)$ as being some sort of 'generalized open cover' of $D(T)$ then we might have a hope that such natural maps of sheaves like the squaring map $\mathcal{O}^\times_{\mathbb{A}^1_\mathbb{C}} \to \mathcal{O}^\times_{\mathbb{A}^1_\mathbb{C}}$ is surjective.

But, pause—why should something like $\operatorname{Spec}(A) \to D(T)$ really be considered a 'generalized open cover'. Well, the answer is actually pretty simple. Namely, it certainly covers (in the sense that $\operatorname{Spec}(A) \to D(T)$ is surjective). Moreover, if we move back to the world of complex manifolds, i.e. we take $\mathbb{C}$-points of the map $\operatorname{Spec}(A) \to D(T)$ and equip these with the complex topology, then this map is a *local isomorphism* or, equivaleantly, it's locally (on the source) an open embedding! In fact, it's a covering map. It's really just the map $\mathbb{C}^\times \to \mathbb{C}^\times$ given by $z \mapsto z^2$. Thus, one reason that we can consider it as a generalized open cover is that, at least on the level of complex points, it's like a 'non-injective open cover' (i.e. it's a covering map, but things get, in this case, double covered).

So, one might wonder whether or not we can make the above example part of a general theory. Namely, is there some notion of 'generalized open cover', as above, such that under this 'topology' (you now see why we use the word 'topology' only intuitively here) many more natural maps of sheaves become surjective which, as the above example shows, might allow us to (through these sheaves) access the (or rather 'a') 'correct topology' of a scheme (one similar to the one we draw, the 'complex one'—*not* the Zariski topology).

2.3 The étale topology and Grothendieck topologies

So, as you might have guessed from the title of the last two subsections, there is a notion of 'generalized open cover' that gives you a reasonable result. What is it? Well…the étale topology! Let me give the rigorous definition.

Let $f : X \to Y$ be a map of schemes. We call f *étale* if it is locally of finite presentation (i.e. for every affine open subscheme $\operatorname{Spec}(B) \subseteq Y$ there exists a cover of $f^{-1}(\operatorname{Spec}(B))$ by affine open subschemes $\operatorname{Spec}(A_i)$ such that the map $B \to A_i$ are finite presentation—this means that, as a B-algebra, A_i is isomorphic to $B[x_1, \ldots, x_n]/I$ for I a finitely generated ideal—not that that n and I depend on i) and one of the following equivalent definition holds:

1. We have that $\Omega^1_{X/Y} = 0$ (this property of f is referred to as being *unramified*) and for all $x \in X$ we have that the map $f_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ (where $y = f(x)$) is flat (this property of f is referred to as being *flat*).

2. f is flat (as in the previous definition) and for all $y \in Y$ the scheme-theoretic fiber $f^{-1}(y) \to \operatorname{Spec}(k(y))$ (i.e. the fiber product $\operatorname{Spec}(k(y)) \times_Y X$ where $\operatorname{Spec}(k(y)) \to Y$ is the inclusion of the residue field of the point y) is isomorphic, as a $\operatorname{Spec}(k(y))$-scheme, to a disjoint union $\displaystyle\bigsqcup_i \operatorname{Spec}(L_i)$ with L_i a finite separable extension of $k(y)$.

3. For all $\operatorname{Spec}(B) \subseteq Y$ affine open, we can find a cover $\{\operatorname{Spec}(A_i)\}$ of $f^{-1}(\operatorname{Spec}(B))$ such that $A_i \cong B[x_1, \ldots, x_n]/I$ (where, again, n and I could depend on i) as a B-algebra where $I = (f_1, \ldots, f_n)$ (note that the n's agree!) such that $\det\left(\dfrac{\partial f_i}{\partial x_j}\right)$ is a unit in A_i.

Exercise 2.2: *Show that the following maps are étale using whatever definition given above you'd like:*

1. *The map $\mathbf{G}_{m,\mathbb{Q}} \to \mathbf{G}_{m,\mathbb{Q}}$ given by $T \mapsto T^2$.*

2. *An open embedding.*

3. *The map* $U \to V$ *where* $V = \operatorname{Spec}(\mathbb{Z}) - \{(2)\}$ *and* $U = \operatorname{Spec}(\mathbb{Z}[i]) - \{(1+i)\}$ *corresponding to the inclusion* $\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathbb{Z}[i][\frac{1}{1+i}]$.

4. *The map* $\operatorname{Spec}(k[x]/(x^n-1)) \to \operatorname{Spec}(k)$ *corresponding to the natural inclusion* $k \hookrightarrow k[x]/(x^n-1)$ *assuming that* $(n, \mathrm{char}(k)) = 1$.

Exercise 2.3: *Show that the following maps are not étale using whatever definition given above you'd like:*

1. *The projection map* $\mathbb{A}^2_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$.

2. *The map* $\operatorname{Spec}(\mathbb{Q}[x,y]/(y^2-x^3)) \to \mathbb{A}^1_{\mathbb{Q}}$ *corresponding to projecting to to the* y*-coordinate.*

3. *The map* $\operatorname{Spec}(k[x]/(x^n-1)) \to \operatorname{Spec}(k)$ *as in Exercise 2.2 number 4. but now assuming that* $\mathrm{char}(k) \mid n$.

We can talk at length some time soon as to *why* this is the correct algebraic notion of 'local isomorphism' or 'generalized open cover', but for now let's suffice ourselves with the following result:

Theorem 2.4: *Let* X *and* Y *be varieties over* $\mathbb{C}$*. Then, a map of* $\mathbb{C}$*-schemes* $f : X \to Y$ *is étale if and only if the map on* $\mathbb{C}$*-points* $X(\mathbb{C}) \to Y(\mathbb{C})$ *(with the complex topologies) is locally (on the source) an open embedding.*

OK, great. Now we have a definition of what a 'generalized open' should mean, but we still don't know technically what a sheaf on this 'generalized topology' is. Let's fix that by giving the following general definition.

Let $\mathscr{C}$ be a category satisfying reasonable conditions we shouldn't worry about (e.g. that it has finite fibered products). Then, a *Grothendieck topology* on $\mathscr{C}$ is a collection $\operatorname{Cov}(X)$ of sets of maps $\{U_i \to X\}$ (this is supposed to be the set of all 'covers' of X) for all objects X of $\mathscr{C}$ satisfying the following conditions:

1. For all X an object of $\mathscr{C}$ the identity map $\{X \xrightarrow{\mathrm{id}} X\}$ is a member of $\operatorname{Cov}(X)$ (i.e. the identity is a covering). More generally, any isomorphism $\{Y \xrightarrow{\approx} X\}$ is in $\operatorname{Cov}(X)$ (i.e. isomorphisms should be coverings)

2. If $\{U_i \to X\}$ is in $\operatorname{Cov}(X)$ and $Y \to X$ is any map, then the set $\{U_i \times_X Y \to Y\}$ is an element of $\operatorname{Cov}(Y)$ (i.e. if $\{U_i \to X\}$ is a cover, and $f : Y \to X$ is a map, then the pullback $\{f^{-1}(U_i) \to Y\}$ is a cover of Y—the fiber product $U_i \times_X Y$ should be thought of as categorical generalization of $f^{-1}(U_i)$).

3. If $\{U_i \to X\}$ is in $\operatorname{Cov}(X)$ and $\{V_{ij} \to U_i\}$ is is in $\operatorname{Cov}(U_i)$ for all i then $\{V_{ij} \to X\}$ obtained by composition is in $\operatorname{Cov}(X)$ (i.e. a covering of a covering gives a covering).

One usually calls a category $\mathscr{C}$ together with a Grothendieck topology a *site*.

Remark 2.5: What we've called here a Grothendieck topology many would call a Grothendieck *pretopology*. You should be nice to such people—they will be your boss some day. But, more seriously, there is a 'better' notion of Grothendieck topology using *cribles* (the French word for 'sieve') but I personally just think it's more complicated, even if it is 'more canonical'. ♦

The point is the following. I said above that if one really thinks about what a sheaf is one really only needs to know what coverings are, not that one needs to know what a topology is. Think about a Grothendieck topology as the minimal notion of coverings necessary to make sheaves work.

To this end we can define a sheaf on a site. Namely, let $\mathscr{C}$ be a site. Then, a *presheaf* on $\mathscr{C}$ with values in the category $\mathscr{D}$ is just a contravariant functor $F : \mathscr{C} \to \mathscr{D}$—so to every object X of $\mathscr{C}$ we obtain an object $F(X)$ in $\mathscr{D}$ and to every morphism $X \to Y$ in $\mathscr{C}$ we obtain a morphism $F(Y) \to F(X)$ in $\mathscr{D}$. Now, while one can define a sheaf in this generality, let's stick to the more comfortable setting of sheaves of sets/abelian groups/groups/rings. Namely, we define a *sheaf* F to be a presheaf on $\mathscr{C}$ such that for all coverings $\{U_i \to X\}$ for X an object of $\mathscr{C}$ (this is where we are using that we have a Grothendieck topology) the following 'sequence is exact':

$$0 \to F(X) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \tag{6}$$

What does this all mean? Well, note that since we have a cover $\{U_i \to X\}$ we get, by functoriality, maps $F(X) \to F(U_i)$. Moreover, note that $U_i \times_X U_j$ (which is thought of as a generalization of $U_i \cap U_j$ since if U_i, U_j are actual opens in a topological space that's what this fibered product is) comes with two projection maps $U_i \times_X U_j \to U_i$ and $U_i \times_X U_j \to U_j$ which in turn induces two maps $F(U_i) \to F(U_i \times_X U_j)$ and $F(U_j) \to F(U_i \times_X U_j)$. The exactness of (6) then means the following the map $F(X) \to \prod_i F(U_i)$ is injective and has image precisely $(t_i) \in \prod_i F(U_i)$ such that for all i, j we have that $t_i \mid_{U_i \times_X U_j} = t_j \mid_{U_i \times_X U_j}$ where we use $t_i \mid_{U_i \times_X U_j}$ to denote the image of $t_i \in F(U_i)$ under the map $F(U_i) \to F(U_i \times_X U_j)$ and similarly for $t_j \mid_{U_i \times_X U_j}$.

Of course, this all just means, interpreted correctly, that for all objects X of $\mathscr{C}$ and all covers $\{U_i \to X\}$ the objects of $F(X)$ are just tuples of elements (t_i) in $\prod_i F(U_i)$ which agree on intersections. In other words, it says that elements uniquely glue! It means F is a sheaf!

Remark 2.6: The fancy way of saying all of this is just to say that $F(X)$ is the equalizer of the diagram

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

in the category $\mathscr{D}$. ♦

As an example of this we have the following:

Exercise 2.7: *Let X be a topological space. Define a category $\mathrm{Open}(X)$ to be the category whose objects are all the open subsets of X and an arrow $U \to V$ (for opens U and V) is just the inclusion (so there is at most one arrow between any two objects). Give $\mathrm{Open}(X)$ the Grothendieck topology where a cover $\{U_i \to U\}$ is literally just a normal open cover. Show that a sheaf on $\mathrm{Open}(X)$ is the same thing as a sheaf on the topological space X in the usual sense.*

So, let us now define, finally, what the (small) étale site of X is—the category of 'generalized opens' in the above desired sense where generalized opens means étale maps. Namely, let us define the *(small) étale site of X*, denote $X_{\text{ét}}$, to have as objects all étale maps $U \to X$, morphisms just normal morphisms over X, and as coverings sets of morphisms $\{U_i \to U\}$ whose only requirement is that the images of the U_i cover U.

Remark 2.8: Why 'small'? It's not because this site has a tendency towards self-deprecation. No, one could easily imagine creating the following site. The underlying objects are <u>all</u> X-schemes $Y \to X$, the morphisms are just morphsims of X-schemes, and the covers $\{U_i \to Y\}$ are étale maps whose images cover Y. This is the *big étale site*. It's like a category whose objects are EVERYTHING but with the 'right' (the étale) covers. But, if we are considering étale maps as generalized opens, then the more natural analogue of $\mathrm{Open}(X)$ for X a topological space is the small étale site, not the big one. The big étale site is more like the following site. Let Top_X denote the site whose objects are all topological spaces Y equipped with continuous maps $Y \to X$, whose morphisms are all continuous maps commuting with the fixed maps to X, and whose coverings are all collections of morphisms $\{U_i \to Y\}$ which are open embeddings that cover Y. Do you see the difference? We will denote the big étale site by $X_{\text{Ét}}$. ♦

As of now it seems hard to show that *anything* is a sheaf on the étale site. Namely, étale maps can be crazy (e.g. see Exercise 2.2), and so it's difficult to imagine how one could really try and show that some given presheaf on $X_{\text{ét}}$ is actually a sheaf. But, thankfully, the following theorem of Grothendieck comes to the rescue:

Theorem 2.9 (Descent for morphisms): *Let Y be an X-scheme. Then, the usual Yoneda embedding $Y \mapsto \mathrm{Hom}_X(-, Y)$ allows one to view Y as a presheaf on $X_{\text{ét}}$. This presheaf is a sheaf.*

So, I want to give you some exercises on sheaves, but first let's get some obvious things out of the way. Namely, let's say that a map of sheaves $\mathcal{F} \to \mathcal{G}$ on $X_{\text{ét}}$ (or any site) is *injective* if for all objects U of $X_{\text{ét}}$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective. We say it's *surjective* if for all U an object of $X_{\text{ét}}$ and $t \in \mathcal{G}(U)$ there is a covering $\{U_i \to U\}$ in $X_{\text{ét}}$ such that $t \mid_{U_i}$ (which, as above, means the image of t under $\mathcal{G}(U) \to \mathcal{G}(U_i)$) is in the image of $\mathcal{F}(U_i)$ for all i. If $\mathcal{F}$ and $\mathcal{G}$ are maps of group/abelian group/ring sheaves we define its *kernel* to be the sheaf associating to U the kernel of the map $\mathcal{F}(U) \to \mathcal{G}(U)$.

Exercise 2.10: *Let X be any scheme with n invertible on X (i.e. n is a unit in $\mathcal{O}_X(X)$). Define the presheaves of abelian groups $\mu_{n,X}$ and $\mathbf{G}_{m,X}$ as we've already discussed. Namely, define them by sending an étale X-scheme U to $\mu_{n,X}(U) = \{f \in \mathcal{O}_U(U)^\times : f^n = 1\}$ and $\mathbf{G}_{m,X}(U) = \mathcal{O}_U(U)^\times$.*

1. *Show that these presheaves are sheaves (you can do this by hand or use Theorem 2.9).*

2. *Show that the sequence*
$$1 \to \mu_{n,X} \to \mathbf{G}_{m,X} \xrightarrow{f \mapsto f^n} \mathbf{G}_{m,X} \to 1 \tag{7}$$
is exact as sheaves (i.e. the map $\mu_{n,X} \to \mathbf{G}_{m,X}$ is injective, the map $\mathbf{G}_{m,X} \to \mathbf{G}_{m,X}$ is surjective with kernel the image of $\mu_{n,X}$).

3. *Show that $\mathbf{G}_{m,X} \to \mathbf{G}_{m,X}$ as above is not necessarily surjective as presheaves (i.e. find an X such that it's not true that for all T one has that $\mathbf{G}_{m,X}(T) \to \mathbf{G}_{m,X}(T)$ is surjective).*

Now, just as you're used to, essentially every operation of sheaves you know that works on topological spaces works for sheaves on sites. In particular, there is a notion of sheafification. Namely, for any presheaf $\mathcal{F}$ on a site there is a map $\mathcal{F} \to \mathcal{F}^s$ where $\mathcal{F}^s$ is a sheaf, and its universal for this property. It's constructed in exactly the same way. Namely, assuming that $\mathcal{F}$ is a separated presheaf (i.e. gluings are unique) which is what will always happen to us, one defines $\mathcal{F}^s(U)$ to essentially be all tuples $(t_i) \in \mathcal{F}(U_i)$ for all coverings $\{U_i \to U\}$ such that the tuples agree on overlaps, and where we identify (t_i) and (s_j) for different coverings $\{U_i \to U\}$ and $\{V_j \to U\}$ if there is a refinement of these covers for which (t_i) and (s_j) restrict to the same tuples.

Remark 2.11: In full disclosure there is a major distinction between sheaves on a classical topological space and sheaves on a general site. Namely, the former has a good notion of 'points' and therefore a good notion of 'stalks'. The notion of points and stalks do exist for general sites (e.g. see this) but, in general, one needn't have 'enough points' to make the notion of stalks useful (e.g. one might not be able to check exactness on stalks). For the étale site this is a non-issue and one can write down all the points explicitly (e.g. see this) but for more exotic topologies, like the flat topology we will later encounter, things can be much more complicated (e.g. see this). ♦

The last thing I want to mention in this section is the notion of quotients and group actions. Namely, suppose that $\mathcal{F}$ is a sheaf of sets on $X_{\text{ét}}$ (or any site) and $\mathcal{G}$ is a sheaf of groups. Then, an *action* of $\mathcal{G}$ on $\mathcal{F}$ is a map of sheaves $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that for all objects U of $X_{\text{ét}}$ the map $\mathcal{G}(T) \times \mathcal{F}(T) \to \mathcal{F}(T)$ is a normal action (note that $\mathcal{G}(T)$ is a group and $\mathcal{F}(T)$ is a set). We then define the *quotient* $\mathcal{F}/\mathcal{G}$ to be the sheafification of the presheaf sending T to $\mathcal{F}(T)/{\sim_{\mathcal{G}}}$ where $\sim_{\mathcal{G}}$ is the usual equivalence relation $t \sim s$ if $t = g \cdot s$ for some $g \in \mathcal{G}(T)$. In other words, it's the sheafification of the orbit presheaf.

As an example of this, note that if $\varphi : \mathcal{G} \to \mathcal{G}'$ is a map of sheaves of groups/abelian groups then can form the quotient presheaf $\mathcal{G}'/\mathcal{G}$ where $\mathcal{G}$ acts on $\mathcal{G}'$ by multiplication through $\mathcal{G}$. If $\mathcal{G}'$ is not a sheaf of abelian groups, then we need that $\varphi(\mathcal{G}(T)) \subseteq \mathcal{G}'(T)$ is a normal subgroup for all T to have that the quotient sheaf $\mathcal{G}'/\mathcal{G}$ is actually a sheaf of groups and not just a sheaf of sets.

Using this we can make the following claim:

Theorem 2.12: *Let $\mathscr{C}$ be a site. Denote by $\mathsf{Ab}(\mathscr{C})$ the category of sheaves of abelian groups on $\mathscr{C}$. Then, $\mathsf{Ab}(\mathscr{C})$ is an abelian category.*

If you don't know what an abelian category is, it just means that all the normal things you do for abelian groups make sense: you have kernels, cokernels, short exact sequences, direct sums, and all of these act in the expected way.

2.4 Cohomology

OK, excellent. So, with the above squared away we can now talk about cohomology. As I said above, cohomology is just some gadget that measures obstructions to things. What obstruction do we have here? Well, suppose that we have a short exact sequence of group sheaves on $X_{\text{ét}}$:
$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{Q} \to 1 \tag{8}$$
where this means that $\mathcal{K} \to \mathcal{G}$ is injective, $\mathcal{G} \to \mathcal{Q}$ is surjective, and $\mathcal{K}$ is the kernel of $\mathcal{G} \to \mathcal{Q}$ (in particular $\mathcal{K}$ is normal). Now, precisely as 3. in Exercise 2.10 shows, just because $\mathcal{G} \to \mathcal{Q}$ is surjective as sheaves it is not true that $\mathcal{G}(T) \to \mathcal{Q}(T)$ is surjective for all T. But, perhaps, we want it to be surjective for some particular purpose—as an example, perhaps we *want* all the elements of $\mathbf{G}_{m,X}(T)$ to have n^{th} roots. This will not happen in general, and so we'd like to figure out to what extent it fails.

This is where sheaf cohomology comes in for the rescue. What is true is that since (8) is exact we know that for all T we have the following sequence is exact

$$1 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \tag{9}$$

with conspicuously missing exactness on the right (the map $\mathcal{G}(T) \to \mathcal{Q}(T)$ is, as mentioned above, not surjective in general). But, while we may not be able to put that extra 1 after $\mathcal{Q}(T)$ and say its exact (because that's what it means to be surjective) perhaps we can put *something* after $\mathcal{Q}(T)$ and say its exact—then how poorly the surjectivity of $\mathcal{G}(T) \to \mathcal{Q}(T)$ fails can be measured by how large this thing is. What is this thing? Cohomology.

Namely, there is an exact sequence from (8) given as follows:

$$1 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \to H^1(T, \mathcal{K}) \to H^1(T, \mathcal{G}) \to H^1(T, \mathcal{Q}) \tag{10}$$

so that the failure of $\mathcal{G}(T) \to \mathcal{Q}(T)$ can be measured, in some sense, by how big $H^1(T, \mathcal{K})$ is. What are these objects $H^1(T, \mathcal{K})$ etc.? Well in this generality they're just pointed sets (meaning sets with a distinguished element). But, this is really because I've insisted talking about sheaves of groups, not sheaves of abelian groups. If we assume that $\mathcal{G}$ is abelian (so that $\mathcal{K}$ and $\mathcal{Q}$ are as well) then the magic of homological algebra (which one can think of as some sort of calculus of abelian categories) says that, in fact, we get a *long exact sequence* (long means long—infinitely extending) sequence from (8) as follows:

$$0 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \to H^1(T, \mathcal{K}) \to H^1(T, \mathcal{G}) \to H^1(T, \mathcal{Q}) \to H^2(T, \mathcal{K}) \to H^2(T, \mathcal{G}) \cdots \tag{11}$$

where now all these H^i things are not just pointed sets, but abelian groups.

Why would you want such a thing? Well, even if you only care about how badly $\mathcal{G}(T) \to \mathcal{Q}(T)$ fails to be exact, the sequence in (10). Specifically, what it means for (10) to be exact is that the image of $\mathcal{G}(T)$ in $\mathcal{Q}(T)$ is the kernel of the map $\mathcal{Q}(T) \to H^1(T, \mathcal{K})$. Let's suppose for a second that $\mathcal{Q}(T) \to H^1(T, \mathcal{K})$ is surjective. Then, this tells us precisely that $\mathcal{Q}(T)/\mathcal{G}(T) \cong H^1(T, \mathcal{K})$ and thus we really do see that $H^1(T, \mathcal{K})$ is measuring the defect of $\mathcal{G}(T) \to \mathcal{Q}(T)$ being surjective. But, what if the map $\mathcal{Q}(T) \to H^1(T, \mathcal{K})$ is <u>not</u> surjective? Well, then we might hope to measure this non-surjectivity using $H^1(T, \mathcal{G})$. And if this map is surjective…I think you see what's going to happen. If we don't know that $\mathcal{Q}(T) \to H^1(T, \mathcal{K})$ is surjective then we're going to repeat the same thing. This repeating process might never stop, and thus really do need all of (11) which really represents the perfect data qualifying how badly $\mathcal{G}(T) \to \mathcal{Q}(T)$ fails to surject.

Since the above long exact sequence thing was due to the magic of homological algebra, the calculus of abelian categories, we can't hope to apply its methods when all of our $\mathcal{G}$, $\mathcal{K}$, and $\mathcal{Q}$ are not abelian—the category of all groups is not an abelian category (having, not shockingly, to do with the fact that all groups are not abelian). One should then see (10) as a kind of unexpected partial extension of (11) to the non-abelian setting (where we have to give up group for pointed set, and our long exact sequence is no longer so long).

OK. Great, so there's this thing called cohomology and it helps us figure out how poorly short exact sequences of sheaves are short exact on particular T (i.e. how poorly the right hand side fails to be surjective). But, an obvious question presents itself: what *are* these cohomology groups? In the abelian case, which is the one you usually learn first, the answer is nice theoretically but somewhat opaque. Namely, one shows that one can resolve all of the groups $\mathcal{G}, \mathcal{K}, \mathcal{Q}$ by 'injective abelian group sheaves' and then use these infinite resolutions (which only exist by abstract existence type arguments—they are <u>not</u> at all explicit) to compute the cohomology group. Very pretty, not very enlightening.

So, one might hope to have a more concrete understanding of these cohomology groups. Since the most important one, since it's most closely tied to our actual question of interest (the failure of surjectivity) and its the only one which works in the non-abelian setting, is H^1, let's focus on that question. Can we give a concrete understanding of what $H^1(X, \mathcal{G})$ means for a group sheaf $\mathcal{G}$?

Yes. We can. Torsors. Namely, while $H^1(X, \mathcal{G})$ in the homological sense, or in the Čech cohomology sense (that we'll mention later), these things are very abstract symbolic things, we can actually identify $H^1(X, \mathcal{G})$ with $\mathcal{G}$-torsors—concrete (often geometric) objects.

2.5 Čech cohomology

Since we're only going to be interested in H^1 in this post, we can do away with most of the fancy machinery of homological algebra and, instead, define things in terms of Čech cohomology. This mitigates the fanciness, but certainly does not super mitigate the opaqueness of the definition (even if it certainly explains why first cohomology is measuring failure of surjectivity).

So, without further adieu, let's define Čech cohomology of an arbitrary abelian group sheaf $\mathcal{G}$ on a site $\mathscr{C}$—in practice we'll mostly be concerned with $X_{\text{ét}}$ and X_{fl} (the latter to be discussed later).

So, Čech cohomology $\check{H}^i(X,\mathcal{G})$ (for X an object of $\mathscr{C}$) will first be defined with respect to an open cover $\{U_i\}$ of X, and then we will obtain the actually cohomology group by taking the limit over all covers.

So, suppose that $\{U_i \to X\}$ is a covering of X in $\mathscr{C}$'s topology. We then define the i^{th} *Čech cohomology group* of $\mathcal{G}$ with respect to this cover, denoted $\check{H}^i(\{U_i\},\mathcal{G})$, as follows. First, consider the following *Čech complex*:

$$0 \to \prod_i \mathcal{G}(U_i) \to \prod_{i,j} \mathcal{G}(U_i \times_X U_j) \to \prod_{i,j,k} \mathcal{G}(U_i \times_X U_j \times_X U_k) \to \cdots \tag{12}$$

What are the maps here? Well, if $(t_i) \in \prod_{i,j} \mathcal{G}(U_i)$ then we map this to the element $(s_{ij}) \in \prod_{i,j} \mathcal{G}(U_{i,j})$ given by $s_{ij} = t_i \mid_{U_i \times_X U_j} - t_j \mid_{U_i \times_X U_j}$. The higher maps are defined similarly (alternating sums of the restrictions). Let us denote the group at position k in (12) by $C^k(\{U_i\},\mathcal{G})$ (where the index k starts at the number -1, and the the group $C^0(\{U_i\},\mathcal{G})$ is $\prod_i \mathcal{G}(U_i)$ so that the term 0 in (12) is $C^{-1}(\{U_i\},\mathcal{G})$). Let us then define for all $k \geqslant 0$

$$\check{H}^k(\{U_i\},\mathcal{G}) := \frac{\ker(C^k(\{U_i\},\mathcal{G}) \to C^{i+1}(\{U_i\},\mathcal{G})}{\operatorname{im}(C^{k-1}(\{U_i\},\mathcal{G}) \to C^k(\{U_i\},\mathcal{G}))} \tag{13}$$

which are the Čech cohomology groups.

Exercise 2.13: *Let $\mathcal{G}$ be a sheaf of abelian groups $\mathscr{C}$.*

1. *Show that $\check{H}^0(\{U_i\},\mathcal{G}) = \mathcal{G}(X)$ for any cover $\{U_i \to X\}$.*

2. *Consider the topological space $X = S^1$ and consider the site* $\operatorname{Open}(X)$ *from above. Consider the open cover $\{U_1, U_{-1}\}$ of X given by $U_1 = S^1 - \{1\}$ and $U_{-1} = S^1 - \{-1\}$. Show that $\check{H}^1(\{U_1,U_{-1}\},\underline{\mathbb{Z}}) \cong \mathbb{Z}$ if $\underline{\mathbb{Z}}$ denotes the constant sheaf on X. This result is actually meaningful, namely the fact that this Čech cohomology group was free abelian of rank one corresponds to the fact that S^1 has one hole.*

3. *Let $X = Y = \mathbf{G}_{m,\mathbb{Q}}$ and let $X \to Y$ be the étale cover defined by $z \mapsto z^2$. Compute the cohomology group $\check{H}^1(\{X \to Y\},\mu_{3,\mathbb{Q}})$.*

OK, so now we want to use the above definition of Čech cohomology of a given cover to define the Čech cohomology of an *object*. Namely, suppose that $\{U_i \to X\}$ is a covering and $\{V_j \to X\}$ is another covering. Say that $\{V_j \to X\}$ *refines* $\{U_i \to X\}$ if there is an open covering $\{W_{ik} \to U_i\}$ for all i such that the open covering $\{W_{ik} \to X\}$ (obtained from the compositions $W_{ik} \to U_i \to X$) is precisely the open cover $\{V_j \to X\}$. In other words, a refinement happens when you cover a cover. Check that if one has a refinement $\{V_j \to X\}$ of $\{U_i \to X\}$ then one gets a map $\check{H}^k(\{U_i\},\mathcal{G}) \to \check{H}^k(\{V_j\},\mathcal{G})$. We then define the *Čech cohomology* of X with respect to $\mathcal{G}$, denoted $\check{H}^k(X,\mathcal{G})$, to be $\varinjlim \check{H}^k(\{U_i\},\mathcal{G})$ as the direct limit ranges over all covers of X (which is filtered by refinement).

So, let us state the following vague result:

Theorem 2.14: *Let $\mathscr{C}$ be any site and $\mathcal{G}$ be any sheaf of abelian groups on $\mathscr{C}$. Then, there are always maps of abelian groups $\check{H}^k(X,\mathcal{G}) \to H^k(X,\mathcal{G})$ (where $H^k(X,\mathcal{G})$ is this fancy homological cohomology group which I didn't define) which is always an isomorphism if $k = 0$ or $k = 1$ and is 'sometimes' an isomorphism if $\mathscr{C}$ and $\mathcal{G}$ are particularly nice.*

Remark 2.15: What's really going on here is there a spectral sequence relating Čech cohomology and derived (the homological algebra) cohomology (e.g. see this). The particularly nice property needed to make the isomorphism hold true for all k happens essentially always when $\mathscr{C}$ is $\operatorname{Open}(X)$ for a topological space X, and not very often for other $\mathscr{C}$—for example for $X_{\text{ét}}$. ♦

So, now, by Theorem 2.14 and our discussion preceding/following (10) we should have, for every short exact sequence of abelian group sheaves as in (8) an exact sequence of groups as in (10). What is this? Namely, what are the maps labeled δ and α,β,γ in the following:

$$1 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \xrightarrow{\delta} \check{H}^1(T,\mathcal{K}) \xrightarrow{\alpha} \check{H}^1(T,\mathcal{G}) \xrightarrow{\beta} \check{H}^1(T,\mathcal{Q}) \tag{14}$$

Well, two of these are easy to define. Namely, we want to define maps $\alpha : \check{H}^1(T,\mathcal{K}) \to \check{H}^1(T,\mathcal{G})$ and $\beta : \check{H}^1(T,\mathcal{G}) \to \check{H}^1(T,\mathcal{Q})$ for any object T of $\mathscr{C}$. It suffices to show that if $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is any map of sheaves of abelian groups, and $\{U_i \to X\}$ is any cover $\mathscr{C}$ then we get an induced map $\check{H}^1(\{U_i\}, mcG_1) \to \check{H}^1(\{U_i\},\mathcal{G}_2)$. But, this is simple. Merely note that for all $i,j,k,\ldots$ we get maps $\mathcal{G}_1(U_i \times_X U_j \times_X U_k \cdots) \to \mathcal{G}_2(U_i \times_X U_j \times_X U_j \times \cdots)$ and thus we evidently get maps $C^k(\{U_i\},\mathcal{G}_1) \to C^k(\{U_i\},\mathcal{G}_2)$. In fact, we get a nice commutative ladder

$$\begin{tikzcd} 0 \arrow[r] & C^0(\{U_i\},\mathcal{G}_1) \arrow[r] \arrow[d] & C^1(\{U_i\},\mathcal{G}_1) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & C^0(\{U_i\},\mathcal{G}_2) \arrow[r] & C^1(\{U_i\},\mathcal{G}_2) \arrow[r] & \cdots \end{tikzcd} \tag{15}$$

and since this ladder commutes, you can show you get induced maps $\check{H}^k(\{U_i\},\mathcal{G}_1) \to \check{H}^k(\{U_i\},\mathcal{G}_2)$ for all $k \geqslant 0$ and, in particular, for $k = 1$. This takes care of α and β.

So, how do we define the map δ? This one is trickier. The idea is the following. Let $t \in \mathcal{Q}(T)$ be arbitrary. Since $\mathcal{G} \to \mathcal{Q}$ was a surjective map of <u>sheaves</u> we can find a cover $\{U_i \to T\}$ such that $t\mid_{U_i}$ is such that there are elements $s_i \in \mathcal{G}(U_i)$ such that the image of s_i under $\mathcal{G}(U_i) \to \mathcal{Q}(U_i)$ is $t\mid_{U_i}$. Now, note that $s_i \mid_{U_i\times_X U_j} - s_j \mid_{U_i\times_X U_j}$ is in $\mathcal{K}(U_i \times_X U_j)$. Why? Well their image under $\mathcal{G}(U_i \times_X U_j)$ is $t\mid_{U_i\times_X U_j} - t\mid_{U_i\times_X U_j} = 0$ and thus their in the image of $\mathcal{K}(U_i \times_X U_j)$ by exactness. One can then show that $(s_i \mid_{U_i\times_X U_j} - s_j \mid_{U_i\times_X U_j})$ is actually in

$$\ker(C^1(\{U_i\},\mathcal{K}) \to C^2(\{U_i\},\mathcal{K})) \tag{16}$$

and that if we had chosen different choices of s_i mapping to $t\mid_{U_i}$ the element of $C^1(\{U_i\},\mathcal{K})$ would differ by an element of

$$\operatorname{im}(C^0(\{U_i\},\mathcal{K}) \to C^1(\{U_i\},\mathcal{K})) \tag{17}$$

and thus from $t \in \mathcal{Q}(T)$ we've obtained an element of $\check{H}^1(\{U_i\},\mathcal{K})$ which then maps to an element of $\check{H}^1(T,\mathcal{K})$. Thus, we've defined our map $\delta : \mathcal{Q}(T) \to \check{H}^1(T,\mathcal{K})$.

The following is routine, but tedious:

Theorem 2.16: *Show that the sequence we've defined as in* (14) *is exact.*

One can imitate the above construction, as mentioned above, to work to give a sequence as in (14) if the $\mathcal{G},\mathcal{K}$ and $\mathcal{Q}$ are any (possibly non-abelian groups)—one needs to use Čech cohomology because homological methods aren't available any more. Let's just take this for granted since, despite some technical difficulties, the idea is the same—we'll atually define $\check{H}^1$ for non-abelian groups below when we discuss torsors.

2.6 Two success stories

While we are going to see ample usages of cohomology soon, let us come back to two examples we discussed earlier. The idea being that we will see some very fruitful uses of sheaves/cohomology on $\mathbb{C}$ (with the complex topology) which is the type of theory we'd like to directly generalize to work on any scheme with the étale topology.

2.6.1 The exponential sequence

Let's start with the exponential sequence. Namely, from (5) we get the following exact sequence of cohomology groups (where everything in sight is happening on the site Open($\mathbb{C}$) where $\mathbb{C}$ is given the 'usual' complex topology)

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to H^1(\mathbb{C},\underline{2\pi i\mathbb{Z}}) \to H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}) \to H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}^\times) \to H^2(\mathbb{C},\underline{2\pi i\mathbb{Z}}) \tag{18}$$

where I've artificially cut off the sequence because this is all that we'll need.

I claimed that one could use the sequence (5) to non-trivially relate analysis/geometry and topology and we are now going to make good on this promise using (18). The key observation is that $H^i(\mathbb{C},\underline{2\pi i\mathbb{Z}})$ is isomorphic to what is called *singular cohomology* $H^i_{\text{sing}}(\mathbb{C},\mathbb{Z})$. I won't define this here (this is the 'cohomology' that one hears algebraic topologists talking about) but mention the following spectacular (although easy to prove!) property of it. Namely, if you can continuously deform a space X into a space Y (rigorously if X and Y are homotopy equivalent) then their singular cohmology groups are the same. Note that $\mathbb{C}$ can be continuously deformed to a point (just contract $\mathbb{C}$ radially from the origin) and thus $H^i_{\text{sing}}(\mathbb{C},\mathbb{Z})$ should be the same as $H^i_{\text{sing}}(\text{pt},\mathbb{Z}) = H^i(\text{pt},\underline{2\pi i\mathbb{Z}})$. The following is then trivial:

Exercise 2.17: *Taking for granted that* $H^i(\text{pt},\underline{2\pi i\mathbb{Z}}) = \check{H}^i(\text{pt},\underline{2\pi i\mathbb{Z}})$ *(this is true for* <u>*any*</u> *manifold) show that* $H^i(\text{pt},\underline{2\pi i\mathbb{Z}}) = 0$ *for all* $i > 0$ *and thus* $H^i(\mathbb{C},\underline{2\pi i\mathbb{Z}}) = 0$ *for all* $i > 0$.

From this and (18) we deduce that $H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}) \cong H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}^\times)$.

Exercise 2.18: *Show, using Čech cohomology and the Mittag-Leffler theorem show that* $H^1(\mathbb{C},\mathcal{O}_C) = 0$.

Thus, we can finally deduce that $H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}^\times) = 0$. Why do we care? Well, $H^1(\mathbb{C},\mathcal{O}_C^\times)$ can be shown (just as in the algebraic case!) to be isomorphic to $\text{Pic}(\mathbb{C})$—the group of holomorphic line bundles on $\mathbb{C}$. Thus, we've shown that every holomorphic line bundle on $\mathbb{C}$ is trivial. But, our method heavily used (5) and (18) to leverage a non-trivial relationship between topology (singular cohomology), analysis ($H^1(\mathbb{C},\mathcal{O}_C) = 0$ is an analytic fact), and complex geometry ($\text{Pic}(\mathbb{C})$).

2.6.2 The 'Kummer' sequence

Let us now analyze another sequence which, while not as powerful as the exponential sequence has a literal, powerful analogue in algebraic geometry. Namely, we saw that since every holomorphic function locally has a square root, that the following exact sequence is exact:

$$1 \to \mu_2 \to \mathcal{O}_{\mathbb{C}}^\times \xrightarrow{f\mapsto f^2} \mathcal{O}_{\mathbb{C}}^\times \to 1 \tag{19}$$

where $\mu_2(U)$ spits out holomorphic functions on U whose square is 1. It's easy to see that we can identify μ_2 with $\underline{\mathbb{Z}/2\mathbb{Z}}$ (the constant sheaf with values in $\mathbb{Z}/2\mathbb{Z}$) since every holomorphic function with square 1 is just a locally constant function with values in $\mu_2(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

So, passing to cohomology we see from (19) that we get a long exact sequence

$$1 \to \mu_2(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^\times(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^\times(\mathbb{C}) \to H^1(\mathbb{C},\underline{\mathbb{Z}/2\mathbb{Z}}) \to H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}^\times) \to H^1(\mathbb{C},\mathcal{O}_C^\times) \to \cdots \tag{20}$$

Again, using that the fact that $H^1(\mathbb{C},\underline{\mathbb{Z}/2\mathbb{Z}}) \cong H^1_{\text{sing}}(\mathbb{C},\mathbb{Z}/2\mathbb{Z})$ we can condense the above, to obtain the following short exact sequence (we truncate the above and take kernels/cokernels):

$$1 \to \mathcal{O}_{\mathbb{C}}^\times(\mathbb{C})/\mathcal{O}_{\mathbb{C}}^\times(\mathbb{C})^2 \to H^1_{\text{sing}}(\mathbb{C},\mathbb{Z}/2\mathbb{Z}) \to \text{Pic}(\mathbb{C})[2] \to 1 \tag{21}$$

(here by $\mathcal{O}_{\mathbb{C}}^\times(\mathbb{C})^2$ we don't mean a product of groups, we mean the squares in $\mathcal{O}_{\mathbb{C}}^\times(\mathbb{C})$) Now, again, we can essentially identify it with the Čech cohomology group on a point) and thus we deduce that $\text{Pic}(\mathbb{C})[2] = 0$ and $\mathcal{O}_{\mathbb{C}}^\times(\mathbb{C})/\mathcal{O}_{\mathbb{C}}^\times(\mathbb{C})^2 = 0$. Now, the first of these we already knew from the previous subsection since we decided there that $\text{Pic}(\mathbb{C}) = 0$ but now we get it 'for free' from topological information since in the previous subsection we had to make the computation $H^1(\mathbb{C},\mathcal{O}_{\mathbb{C}}) = 0$ and now we need do nothing. We also see that every nowhere vanishing entire function on $\mathbb{C}$ is automatically a square. While this is not overly complicated using complex analysis, we get it here *for free* using the topology of $\mathbb{C}$. But, such a connection (between the analysis and the topology) would never have been possible without sheaves, an ample number of short exact sequences to relate them (which was directly related to how fine the topology of $\mathbb{C}$ was), and cohomology.

3 Torsors

Now, with all of the above said I hope you can appreciate two things:

1. Cohomology is important to understand connections between sheaves, especially the more mysterious sheaves we've created by introducing the étale topology.

2. Cohomology groups are somewhat opaque. Namely, even though we've defined them in a very explicit manner (using Čech theory) it's not at all clear what a cohomology class 'means'—what it 'looks like'.

We would now like to use torsors to clear this up a bit.

3.1 Quick recollection on group schemes

Before we jump straight into torsors, it will be helpful to have some base knowledge about group schemes. Most of this might be well-known to you already, and if so you can skip it.

The idea, not shockingly, is that a group scheme should be a scheme and a group in such a way that the two structures play nicely together—in more highfalutin language a group scheme is a group object in the category of schemes. Specifically, let X be a fixed scheme. We then define a *group scheme over X* or an *X-group* to be an X-scheme $G \to X$ equipped with maps $m : G \times_X G \to G$, $i : G \to G$, and $e : X \to G$ which satisfy all the usual diagrams for a group with m the multiplication map, i the inversion map, and e the identity (section).

Remark 3.1: Be careful! A group scheme is a scheme and so, in particular, is a set (a scheme is a topological space with a sheaf of rings such that...). But, the underlying set of G is not a group. The issue is that if the underlying set of G were to have a group structure we'd have a map $m : G \times G \to G$ where the $\times$ here is set product. But, the underlying set of $G \times_X G$ is not $G \times G$! This can be a bit annoying, but is made somewhat more palatable by the below discussion.

The more correct idea is that a group scheme is like a *family* of groups over X. Namely, note that our multiplication is from $G \times_X G$ to G. What are points of $G \times_X G$? Well, intuitively they are pairs $(g, h) \in G \times G$ such that g and h map to the same point of X. In other words, our multiplication map is only telling us how to multiply points of G in the same fiber of the map $G \to X$. So, for every point $x \in X$ we get a group G_x (the base change of G over the map $\mathrm{Spec}(k(x)) \to X$—the fiber) and G is something like $\{G_x\}_{x\in X}$. Even then the groups G_x can't be really thought of as groups in the traditional sense if $k(x)$ is not algebraically closed. If $k(x)$ is algebraically closed then G_x can be essentially thought of as $G(k(x))$ (in the same way that in Chapter 1 of Hartshorne he identifies varieties over an algebraically closed field k with their set of k-points) and $G(k(x))$ is an honest-to-god group. ♦

A much more useful way of thinking about group schemes comes from the Yoneda perspective. Namely, let's recall that sending $Y \mapsto \mathrm{Hom}_X(-, Y)$ defines a fully faithful embedding $\mathsf{Sch}/X \hookrightarrow \mathsf{PSh}(X)$ where, here, Sch/X is the category of X-schemes and $\mathsf{PSh}(X)$ is the category of presheaves (i.e. contravariant Set valued functors) on Sch/X. We will almost always abbreviate $\mathrm{Hom}_X(T, Y)$, for T an X-scheme, to just $Y(T)$. Moreover, if T is affine, say $\mathrm{Spec}(R)$, we are likely to abbreviate $Y(\mathrm{Spec}(R))$ further to $Y(R)$. Because of Yoneda one commonly (if a bit sloppily) conflates Y with the functor $\mathrm{Hom}_X(-, Y)$—so when I say 'Y an X-scheme' I mean both the actual scheme Y (the locally ringed space with a map to X) and its associated presheaf on Sch/X.

Then, we have the following basic results:

Theorem 3.2: *Let G be an X-scheme. Then, to give G the structure of an X-group is equivalent to give a factorization of $G : \mathsf{Sch}/X \to \mathsf{Set}$ through the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$.*

In less fancy words this says that to give G the structure of an X-group is to give a group structure on $G(T)$ functorial in T (i.e. such that if $T \to S$ is a map of X-schemes the induced map $G(S) \to G(T)$ is a map of groups).

Exercise 3.3: *Give a proof of Theorem 3.2.*

With Theorem 3.2 we can give a multitude of group schemes over X (any scheme). For example, we have the *general linear group* $\mathrm{GL}_{n,X}$ over X which assigns to an X-scheme T the group $\mathrm{GL}_n(\mathcal{O}_T(T))$ (where, recall, the group $\mathrm{GL}_n(R)$ are $n \times n$ invertible matrices with entries in R). We have the *special linear group* $\mathrm{SL}_{n,X}$ assigning each X-scheme T to $\mathrm{SL}_n(\mathcal{O}_T(T))$ (where $\mathrm{SL}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ with determinant 1). We have $\mathbf{G}_{m,X}$ which is called the *multiplicative group* and is just $\mathrm{GL}_{1,X}$. We have the *additive group* $\mathbf{G}_{a,X}$ which sends T to $\mathcal{O}_T(T)$ (as an additive group). And, as a final example, we have the *n^{th}-roots of unity* over X, denoted $\mu_{n,X}$, which sends an X-scheme T to $\{f \in \mathcal{O}_T(T)^\times : f^n = 1\}$.

Exercise 3.4: *The above examples are really, a priori, examples of group sheaves not group schemes. I didn't prove they were representable. Try and show they are representable by writing down schemes G over X realizing the above group sheaves.*

Exercise 3.5: *One must be careful that in Theorem 3.2 there is nothing that says you can't factor the functor G : $\mathsf{Sch}/X \to \mathsf{Set}$ through Grp in more than one way. Find an example of an X-scheme G with more than one non-isomorphic group structure.*

A map of X-groups is a map of X-schemes commuting with the multiplication maps. By Theorem 3.2 this is the same thing as a map of their associated group sheaves.

We will have need, since we want to be rigorous, to talk about various types of group schemes that will be important to us. So, let us say that an X-group G is a *affine* if the underlying X-scheme is an affine X-scheme (recall this does not mean that G is affine, just that the map $G \to X$ is affine) and we call it an *algebraic X-group* if it is affine and of finite presentation. All of the examples we gave above are algebraic X-groups. A non-example is given by elliptic curves (or abelian vareities) which, while finite presentation group schemes, are not affine.

We will also need to know what it means for G to be smooth, so let us define what it means for a map of schemes, in general, to be smooth. Namely, let $f : X \to Y$ be a map locally of finite presentation. We call f *smooth* if for all $y \in Y$ there exists opens $y \in V \subseteq Y$ and $U \subseteq X$ with $f(U) \subseteq V$ and a factorization

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & \mathbb{A}^d_V \arrow[d] \\ & V \end{tikzcd} \tag{22}$$

for some positive integer d with $U \to \mathbb{A}^d_v$ étale and $\mathbb{A}^d_V \to V$ the natural projection map.

Remark 3.6: Remember that smooth maps are supposed to be like submersions, and so the above is an 'étale local' version of the submersion theorem. ♦

We then call an X-group G *smooth* if its underlying X-scheme is smooth (i.e. the map $G \to X$ is smooth).

Exercise 3.7: *All of the examples of group schemes we gave above was smooth save one whose smoothness depends on whether or not n is invertible in $\mathcal{O}_X(X)$. Which one is it?*

The following result will tell us, in most cases, that we don't really have to worry about any of this:

Theorem 3.8 (Chevalley): *Let k be a field of characteristic 0, and G a $\operatorname{Spec}(k)$-algebraic group. Then, G is automatically smooth.*

3.2 Torsors as sheaves

We begin with what is, undoubtedly, the less 'enlightening' perspective of torsors (at least from a geometric perspective) but the one that is easier to manipulate formally. What's the idea? Well, if $\mathcal{G}$ is a group sheaf on a site $\mathscr{C}$ then there is, amongst all $\mathcal{G}$-sheaves (i.e. sheaves of sets on $\mathscr{C}$ with an action by $\mathcal{G}$), a *canonical one*. Namely, $\mathcal{G}$ acts on itself by left translation. A $\mathcal{G}$-torsor is then a 'twist' of this—a $\mathcal{G}$-sheaf locally (for the topology on $\mathscr{C}$) isomorphic to this canonical one.

So, to this end, let us make the following definition. Let $\mathcal{G}$ be a group sheaf on $\mathscr{C}$. Then a *$\mathcal{G}$-torsor* is a $\mathcal{G}$-sheaf $\mathcal{F}$ satisfying the following two conditions:

1. For all X an object of $\mathscr{C}$ there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}(U_i) \neq \varnothing$ for all i.
2. If $\mathcal{F}(X)$ is non-empty, then the action of $\mathcal{G}(X)$ on $\mathcal{F}(X)$ is simply transitive.

We then have the following theorem justifying our original discussion:

Theorem 3.9: *Let $\mathcal{F}$ be a $\mathcal{G}$-sheaf. Then, $\mathcal{F}$ is a $\mathcal{G}$-torsor if and only if for all X an object of $\mathscr{C}$ there exists a cover $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic, as a $\mathcal{G}|_{U_i}$-sheaf to $\mathcal{G}|_{U_i}$ acting on itself by left multiplication (see below for notation).*

Exercise 3.10: *Prove Theorem 3.9.*

Remark 3.11: Since we haven't defined it above, let me mention the following. For any site $\mathscr{C}$ and any object X of $\mathscr{C}$ we obtain a new site $\mathscr{C}/X$ whose objects are X-objects of $\mathscr{C}$ (i.e. objects Y of $\mathscr{C}$ with equipped maps $Y \to X$), whose morphisms are morphisms over X, and whose covers are the same as those of $\mathscr{C}$ (i.e. a cover of $\{Y\}$ in $\mathscr{C}/X$ is an element $\{U_i \to Y\}$ of $\mathrm{Cov}(Y)$ for $\mathscr{C}$ with each $U_i \to Y$ an X-morphism).

If $\mathcal{F}$ is any sheaf on $\mathscr{C}$ then by restricting $\mathcal{F}$ to the subcategory $\mathscr{C}/X$ we obtain a new sheaf. We denote this sheaf by $\mathcal{F}|_X$. ♦

Torsors are somewhat difficult to give non-silly examples of (an indication of their depth!), and so let us postpone our discussion of examples until §3.

So, let us define a *morphism* of $\mathcal{G}$-torsors $f : \mathcal{F}_1 \to \mathcal{F}_2$ to be just a morphism of sheaves on $\mathscr{C}$ which commutes with the $\mathcal{G}$-action. We then have the following basic result:

Theorem 3.12: *Every morphism of $\mathcal{G}$-torsors is an isomorphism.*

Exercise 3.13: *Prove Theorem 3.12.*

Remark 3.14: This is a somewhat surprisingly deep result, not in proof, but in usage. Namely, it tells one that torsors are amenable to study by something called 'stacks'. This at least shows that the classifying stack BG (where G is an algebraic X-group) is a category fibered in groupoids. ♦

Let us now create the following two definitions. Namely, let $\mathbf{Tors}(\mathcal{G})$ (or $\mathbf{Tors}(\mathcal{G}, \mathscr{C})$ when we want to empahsize the dependence on the site $\mathscr{C}$) denote the category of $\mathcal{G}$-torsors over $\mathscr{C}$ and let $\mathrm{Tors}(\mathcal{G})$ (or $\mathrm{Tors}(\mathcal{G}, \mathscr{C})$ with the same comment) denote the set of *isomorphism classes* of $\mathcal{G}$-torsors. If $\mathcal{F}$ is an object of $\mathbf{Tors}(\mathcal{G})$ we denote its image in $\mathrm{Tors}(\mathcal{G})$ (i.e. its isomorphism class) by $[\mathcal{F}]$.

Now, the set $\mathrm{Tors}(\mathcal{G})$ is a *pointed set*—in other words, it has a distinguished element. This corresponds to the isomorphism class of the *trivial $\mathcal{G}$-torsor*: $\mathcal{G}$ (with left multiplication). We have the following basic result which identifies the trivial $\mathcal{G}$-torsor structurally:

Exercise 3.15: *Suppose that $\mathscr{C}$ has a final object X_0 (so that $\mathscr{C} = \mathscr{C}/X_0$). Then, show that $\mathcal{F}$, an object of $\mathbf{Tors}(\mathcal{G})$, is isomorphic to the trivial $\mathcal{G}$-torsor if and only if $\mathcal{F}(X_0) \neq \varnothing$.*

Torsors are also a functorially defined object. Namely, various maps between objects involved in the definition of torsor define various maps between categories of torsors. We focus here on the functoriality in the group sheaf (opposed to the functoriality in the site). To this end, suppose that $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is a map of group sheaves on $\mathscr{C}$. We then want to define a functor $\mathbf{Tors}(\mathcal{G}_1) \to \mathbf{Tors}(\mathcal{G}_2)$ which will, in turn, define a map of pointed sets $\mathrm{Tors}(\mathcal{G}_1) \to \mathrm{Tors}(\mathcal{G}_2)$.

To this end, for a $\mathcal{G}_1$-torsor $\mathcal{F}$ let us define the *contracted product* $\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{F}$ as follows. Consider the product sheaf $\mathcal{G}_2 \times \mathcal{F}$ defined, as one would expect, as sending X in $\mathscr{C}$ to $\mathcal{G}_2(X) \times \mathcal{F}(X)$. Define an action of $\mathcal{G}_1$ on $\mathcal{G}_2 \times \mathcal{F}$ as follows. For all X defined $\mathcal{G}_1(X)$'s action on $\mathcal{G}_2(X) \times \mathcal{F}(X)$ by declaring that $g_1 \cdot (g_2, f) = (g_2\varphi(g_1)^{-1}, g_2 \cdot f)$ utilizing here the homomorphism φ and the action of $\mathcal{G}_1$ on $\mathcal{F}$. The contracted product $\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{F}$ is then the quotient $(\mathcal{G}_2 \times \mathcal{F})/\mathcal{G}_1$. We might alternatively call this the *induced torsor* from φ and denote it $\varphi_*(\mathcal{F})$.

Note that we get a natural action of $\mathcal{G}_2$ on $\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{F}$ which, on the level of the presheaf (which then passes to the sheaf since $\mathcal{G}_2$ is a sheaf), acts on X-points by $h \cdot [(g_2, f)] := [(hg_2, f)]$ for any equivalence class $[(g_2, f)] \in (\mathcal{G}_2(X) \times \mathcal{F}(X))/\mathcal{G}_1(X)$. One easily checks that this is a well-defined action.

As one would hope, we have the following result:

Theorem 3.16: *The $\mathcal{G}_2$-sheaf $\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{F}$ is a $\mathcal{G}_2$-torsor, and the association $\mathcal{F} \mapsto \mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{F}$ is naturally a functor $\mathbf{Tors}(\mathcal{G}_1) \to \mathbf{Tors}(\mathcal{G}_2)$. Moreover, the image of the trivial torsor class $[\mathcal{G}_1]$ is the trivial torsor class $[\mathcal{G}_2]$.*

Exercise 3.17: *Prove Theorem 3.16.*

By Theorem 3.16 we know that the operation $[\mathcal{F}] \mapsto [\mathcal{G}_2 \times^{\mathcal{G}_1} \mathcal{F}]$ is a well-defined map of pointed sets $\mathrm{Tors}(\mathcal{G}_1) \to \mathrm{Tors}(\mathcal{G}_2)$ which we denote φ_*.

Finally, we would like to discuss how the pointed set $\mathrm{Tors}(\mathcal{G})$ inherits the structure of an abelian group when $\mathcal{G}$ is not just a sheaf of groups but a sheaf of *abelian groups* (sometimes called an *abelian sheaf*). In particular, suppose that $\mathcal{G}$ is an abelian sheaf and $\mathcal{F}_1$ and $\mathcal{F}_2$ are objects of $\mathbf{Tors}(\mathcal{G})$. Let us then define the *sum*$[\mathcal{F}_1] + [\mathcal{F}_2]$ to be $[\mathcal{F}_3]$ where $\mathcal{F}_3$ is defined as the quotient sheaf $(\mathcal{F}_1 \times \mathcal{F}_2)/\mathcal{G}$ where $\mathcal{G}$ acts on $\mathcal{F}_1 \times \mathcal{F}_2$ on T-points by having $g \cdot (f_1, f_2) := (gf_1, g^{-1}f_2)$. Note that $\mathcal{G}$ acts on this quotient via its action on the presheaf quotient given by $g \cdot [(f_1, f_2)] = [(gf_1, f_2)] = [(f_1, gf_2)]$ if square brackets denote the class in the quotient set.

Exercise 3.18: *Show that if $\mathcal{G}$ is an abelian sheaf on $\mathscr{C}$, then* Tors($\mathcal{G}$) *with the above operation gives a well-defined abelian group structure on* Tors($\mathcal{G}$) *with the distinguished element as the identity element, and the inverse given by having $\mathcal{G}$ act via the opposite action (i.e. $g \cdot^{opp} f := g^{-1} \cdot f$).*

3.3 Torsors as spaces

Now that we have gotten our fill of the theory in terms of sheaves which, again, is cleaner if less geometrically pleasing, we would now like to recast the theory in terms of spaces instead of sheaves.

Now, while essentially all of our torsors will be 'étale torsors' it's technically easier to initially work in a finer topology than that of the étale topology.

3.3.1 The flat topology

So, to motivate why we would ever want to go to an even *finer* topology than the étale topology, consider the following example. We discussed in Exercise 2.10 part 2. that under favorable condition we have an exact sequence of sheaves on $X_{\text{ét}}$ (or the big étale site) given by the following:

$$1 \to \mu_{n,X} \to \mathbf{G}_{m,X} \to \mathbf{G}_{m,X} \to 1 \tag{23}$$

But, for arithmetic reasons (e.g. when trying to prove class field theory or the Mordell-Weil theorem) it's often times extremely useful to wok with the sequence (23) when $X = \operatorname{Spec}(\mathbb{Z})$. We then run into a serious issue: the sequence is no longer exact on the big étale site of Spec($\mathbb{Z}$). The issue is that the cover one wants to take to obtain, say, $2 \in \mathbf{G}_{m,X}(\operatorname{Spec}(\mathbb{Z}) - \{(2)\}$ as an n^{th}-root is $\operatorname{Spec}(\mathbb{Z}[\sqrt[n]{2}]) \to \operatorname{Spec}(\mathbb{Z})$ and this is <u>not</u> étale! In fact, while one has the sequence (23) is actually a short exact sequence on the *finite étale site* (defined as one might imagine) when n is invertible in $\mathcal{O}_X(X)$ the situation is totally untenable for Spec($\mathbb{Z}$)—it has *no* (non-trivial connected) finite étale covers!

Remark 3.19: This complication is in no small part related to the fact that μ_n is *never* smooth over Spec($\mathbb{Z}$) if $n > 1$.♦

Remark 3.20: To see a rigorous justification of non-exactness one can see this. ♦

For reasons like this, while the étale topology is usually well-suited to geometric (in the sense of the study of varieties) questions, more fine-structured arithmetic questions often require one to work with a finer topology. This topology is largely untenable (without recourse to the more powerful geometric étale topology) but is extremely helpful as a technical tool—even though we don't want to work with this larger topology sometimes, it is certainly helpful to know that in *some* describable topology essentially all sequences that 'should be' exact, are exact. It also has the added theoretic benefit of computational ease (although not in practice) by allowing one to have more covers and thus, consequently, a greater chance at a Leray cover.

Remark 3.21: We said in Theorem 2.14 that under reasonable hypotheses $H^k(X,\mathcal{G}) \cong \check{H}^k(X,\mathcal{G})$. A (finite) *Leray cover* is a (finite) cover $\{U_i \to X\}_{i=1,\ldots,n}$ such that $H^i(U_I,\mathcal{G}) = 0$ for all $i > 0$ and all $|I| \geqslant 1$ (here for a subset $I = \{i_1,\ldots,i_\ell\} \subseteq \{1,\ldots,n\}$ we denote by U_I the object $U_{i_1} \times_X \cdots \times_X U_{i_\ell}$). Such covers have the incredible property that $H^i(X,\mathcal{G}) \cong \check{H}^i(X,\mathcal{G}) \cong \check{H}^i(\{U_i\},\mathcal{G})$—thus one really only has to worry about computing cohomology on this single cover opposed to worrying about all possible covers.

Thus, the comment above about the computational niceness of a finer topology alludes to the fact that more covers give a greater chance of finding a Leray cover. Of course, this is only theoretically true—in practice finer cohomology theories are usually more difficult to compute with. ♦

OK. So, what is this larger topology? Let us define a morphism of schemes $f : X \to Y$ to be *fppf* (the acronym for the French phrase '*fidèlement plat et présentation finie*' which translated means 'faithfully flat and finite presentation') if it is flat, surjective, and locally of finite presentation. We then define the *small flat site* X_{fl} of X (also called the *small fppf site*) as follows. Let us define a *flat covering* $\{Y_i \to X\}$ of schemes to be a collection of flat finitely presented morphisms $Y_i \to X$ such that $\bigsqcup_i Y_i \to X$ is fppf. The objects of X_{fl} are then morphisms $Y \to X$ belonging to some flat covering of X, the morphisms are X-morphisms and the coverings are flat coverings (as X-schemes). The *big flat site* X_{Fl} has as its underlying category the category Sch/X, as its morphisms X-morphisms, and as its covers flat coverings (as X-schemes).

Exercise 3.22: *Show that the sequence* (23) *is always exact as a sequence of sheaves on* X_{Fl}.

Remark 3.23: One might ask why the flat topology is the 'natural one' to work in—why is it a next natural choice for 'generalized opens' when the étale topology is too coarse? Well, I don't have a great answer for this other than *it works*—most reasonable objects are flat.

In general, given any category $\mathscr{C}$ one can always try and equip $\mathscr{C}$ with the finest topology which is 'reasonable'. What does reasonable mean here? Well, putting a topology on $\mathscr{C}$ is a lot like defining what presheaves on $\mathscr{C}$ are sheaves (this is not literally true—the topos doesn't determine the site—but let's not get into this here) and there is always a natural class of presheaves on $\mathscr{C}$ that one would want to be sheaves—the representable ones. This roughly says that whatever notion of topology we define on $\mathscr{C}$ it should be coarse enough to be able to glue maps uniquely on covers. Let's say that a topology on $\mathscr{C}$ is *subcanonical* if every representable presheaf is a sheaf.

Not shockingly, we might then try to consider the *finest* topology on $\mathscr{C}$ which is subcanonical and, again not shockingly, this topology exists and is called the *canonical topology.* So, in the 'choice free' sense of the world canonical the most canonical topology on Sch/X is the canonical topology. But, this is so huge, so overwhelmingly fine as to be difficult to deal with in practice (they are the so-called universally strict epimorphisms).

So, the flat topology on Sch/X is subcanonical (i.e. it's reasonable—this is also not obvious, it's part of what's called *fppf descent*) but it is not the canonical topology. Think of it has a sort of happy medium between large enough to encompass the exactness of most sequences we care to be exact, but small enough to be potentially usable. There is, in fact, an even *finer* topology that comes up in practice called the *fpqc*, the '*fidèlement plat et quasi-compact*' topology, which does have some marked improvements over the fppf topology. For one, in the fppf topology the map $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ is (almost always) not a cover, where this is the case for the fpqc topology (this is because the map is <u>not</u> finite presentation in general, but is always quasi-compact). Let us not get overly obsessed with this minutiae here. ♦

3.3.2 Flat torsors and principal bundles

Fix a scheme X and a group sheaf $\mathcal{G}$ on X_{Fl}. Let us then define, as we should, a *flat torsor* on X to be a $\mathcal{G}$-torsor on the site X_{Fl}. What makes these torsors nice is the fact that they are often representable if $\mathcal{G}$ is.

To this end, let's now assume that G is a flat algebraic X-group (which, as usual, we identify with its associated group sheaf). We define a *principal G-bundle* (or *principal homogenous space* for G) to be a flat finite presentation X-scheme $f : Y \to X$ with an action of G satisfying the following equivalent properties:

1. The morphism $Y \times_X G \to Y \times_X Y$ defined on T-points by sending $(y, g) \in Y(T) \times G(T)$ to (y, gy) is an isomorphism of X-schemes.

2. There exists an open covering $\{U_i \to X\}$ in X_{fl} such that Y_{U_i} is isomorphic, as a G_{U_i}-space, to G_{U_i} with its left multiplication action.

where, here, G_{U_i}-space just means a U_i-scheme with a G_{U_i} action. These are an algebro-geometric analogue of principal bundles in topology/differential geometry and so are rife with geometric intuition—I will not recall this intuition here since there are ample sources doing so in the topology/differential geometry literature.

Note also that this looks startlingly familiar to the definition of torsor, and this is not by accident. Before making this precise, let us define a morphism of principal G-bundles to be a morphism of X-schemes commuting with the G-action.

We then have the following:

Theorem 3.24: *The morphism sending $Y \mapsto \mathrm{Hom}_X(-, Y)$ is an equivalence of categories from the category of principal G-bundles to the category of G-torsors on X_{Fl}. Similarly, the morphism sending Y to $\mathrm{Hom}_X(-, Y)$ to the category of G-torsors on X_{fl} is an equivalence.*

Remark 3.25: Let us note that the assumption that G was a flat algebraic X-group was *pivotal* here. In particular, the real important assumption was that G was X-affine. In fancy language, that is because *every* G-torsor for *any* group scheme G is an *algebraic space* (in the sense of Artin). But, an algebraic space which is locally affine can be shown to actually be a scheme. Since we assumed our G was affine, and a G-torsor is locally isomorphic go G, we conclude that our algebraic space is actually locally affine and thus a scheme. ♦

Thus, for the site X_{Fl} we can concretely understand torsors as honest-to-god spaces over X (at least when $\mathcal{G}$ is representable by a flat algebraic X-group). This will make the discussion of such objects much more satisfying/pleasing as well as easier to compute (in the sense that knowing we're looking for spaces is often psychologically easier).

Let us give a somewhat subtle corollary to Theorem 3.24 which allows us to be cavalier with working in X_{Fl} versus X_{fl}:

Corollary 3.26: *Let X be a scheme. Then, there is a natural equivalence $\mathbf{Tors}(X_{\mathrm{fl}}, G) \cong \mathbf{Tors}(X_{\mathrm{Fl}}, G)$ inducing a bijection of pointed sets $\mathsf{Tors}(X_{\mathrm{fl}}, G) \xrightarrow{\approx} \mathsf{Tors}(X_{\mathrm{Fl}}, G)$ which is an isomorphism of abelian groups if G is abelian.*

Thus, while one might be concerned that there is some big difference between working in X_{Fl} and X_{fl} there is none whatsoever—we will then choose to work in X_{Fl} because of the simpler description of its underlying category.

So, now, as an example of a statement for torsors (as sheaves) which is more pleasing geometrically is the following:

Exercise 3.27: *Let $Y \to X$ be a principal G-bundle for the flat algebraic X-group G. Then, show that $Y \to X$ is the trivial principal G-bundle if and only if $Y \to X$ has a section.*

3.3.3 G-torsors and quotients

Let us end this subsection by discussing, in a very informal sense, how torsors are related to 'quotients'. Whereas most of the above sections were trying to be semi-rigorous, you should treat this subsection purely as a for-fun bonus section since it would take far too long to make anything here rigorous.

As it turns out, quotients are a preternaturally sticky subject in algebraic geometry and *a lot* of powerful minds have spent *a lot* of time formulating various notions of what 'quotient' means, and when they exist. For cultural enrichment let me mention two here:

1. Mumford, most notably in this famous text *Geometric Invariant Theory*, attempted to understand quotients in a down-to-earth method well-suited to studying the action of reductive group varieties on proper varieties (e.g. flag varieties).

2. The theory of algebraic stacks and algebraic spaces had the great triumph of being able to canonically realize the 'quotient stack' $[X/G]$ for, say, a smooth group G acting on a scheme X.

To get a minuscule idea of what type of difficulties arise in even defining the 'right notion' of quotient, consider the following. Let $X := \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[T])$ and let G be $\underline{\mathbb{Z}/n\mathbb{Z}}$ (the constant group over $\mathbb{C}$ with values in $\mathbb{Z}/n\mathbb{Z}$) act on X by having $k \mod n$ act as $T \mapsto \zeta^k T$ where ζ is some fixed primitive n^{th}-root of unity. In other words, we let $\underline{\mathbb{Z}/n\mathbb{Z}}$ act on X by multiplication by roots of unity. What then should X/G be? Well, whatever it should be, the map $X \to X/G$ should be finite and thus, by Chevalley's theorem, X/G should be affine. What then should the ring of functions on X/G be—in other words, X/G is the spectrum of what ring?

Well, one might imagine that maps $X/G \to Y$, for some $\mathbb{C}$-variety Y, should be G-equivariant maps $X \to Y$. Thus, a reasonable guess is that $X/G = \mathrm{Spec}(\mathbb{C}[T]^G)$—the functions on the quotient X/G should be the G-equivariant functions on X. Thinking about this for a second shows that this would mean that $X/G = \mathbb{A}^1_{\mathbb{C}}$ and the map $X \to X/G$ is just the n^{th}-power map $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. This has certainly missed something—every point of X except 0 has trivial isotropy by G whereas 0 has all of G as an isotropy subgroup. You'd expect that the quotient X/G should reflect this fact by having 0 'count G times'. Thus you might expect, if anything, that X/G is non-reduced with the image of 0 being a fat point of degree n. But, no, that does not happen here.

The issue is that the quotient should look like $X/G = \mathbb{A}^1_{\mathbb{C}}$ except at 0 there should be extra structure recording that 0 in X had isotropy group G. In fact, we could do this for all points of X/G. They should really be points (Gx, G_x) where Gx is an orbit and G_x is an isotropy subgroup.

Here's a more striking example (in the complex analytic category) of this that you might be more familiar with if you've studied number theory. Namely, let $\mathfrak{h}$ denote the upper half-plane. Then, one can show that $\mathfrak{h}$ has a very nice 'moduli description'. Namely, it parameterizes pairs (E, α) where E is an elliptic curve and α is an (orientation preserving) trivialization of the singular homology $H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2$ (singular homology is the dual of singular cohomology which is the (Čech) cohomology of the constant sheaf $\underline{\mathbb{Z}}$). What I mean by parameterizes is that one can make sense of the data (E, α) in families (if this means anything to you by such a pair (E, α) over a complex manifold X we mean an elliptic curve $f : E \to X$ which roughly means a holomorphic map whose fibers are all elliptic curves in the usual sense, and an (orientation preserving) isomorphism $(R^1 f_* \underline{\mathbb{Z}})^\vee \cong \underline{\mathbb{Z}}^2)$) and that, in fact, $\mathfrak{h}$ represents this data: there is a functorial identification:

$$\mathrm{Hom}(X, \mathfrak{h}) \cong \{(E, \alpha)\} / \approx \tag{24}$$

where $\approx$ means that we identify equivalent pairs (which means what you think it means—isomorphisms of the E's that carries one α to the other).

Remark 3.28: To see a rigorous discussion of the above see §6 of this. ♦

But, perhaps we are more interested in not studying elliptic curves with a trivialization of their homology but, instead, *just elliptic curves*. There is a natural way we might do this. Namely, we want to go from classifying pairs (E, α) to just classifying E's. If we can find an action of some group G on $\mathfrak{h}$ such that (E, α) and (E', α') are in the same orbit if and only if $E = E'$ (so that an orbit is just identifying different choices of α on a fixed E) then we should be able to define some space that parameterizes just elliptic curves by considering $\mathfrak{h}/G$.

Well, we can find a group acting on $\mathfrak{h}$ that does precisely this—it's $\mathrm{SL}_2(\mathbb{Z})$ acting by fractional linear transformations. So, we'd hope that we could parameterize all elliptic curves by the space $\mathfrak{h}/\mathrm{SL}_2(\mathbb{Z})$. But, again, this is riddled with issues related to the fact that $\mathfrak{h}$ has lots of points with non-trivial isotropy subgroups. And, indeed, if one tries to give $\mathfrak{h}/\mathrm{SL}_2(\mathbb{Z})$ *some structure* as a complex manifold what you end up with is just $\mathbb{A}^1_{\mathbb{C}}$ with the map $\mathfrak{h} \to \mathbb{A}^1_{\mathbb{C}}$ essentially sending (E, α) to the j-invariant $j(E) \in \mathbb{C}$. This clearly has not been a very faithful notion of quotient since the j-invariant certainly does *not* classify elliptic curves in families—$\mathbb{A}^1_{\mathbb{C}}$ does not parameterize elliptic curves (up to isomorphism) in families. Again, we should somehow be keeping track of the isotropy subgroups to get a more faithful notion of quotients.

So, we need a notion of space that not only has points, but groups attached to those points. And, while this is a occasionally decried heuristic, we can think very roughly that the objects that are schemes with groups attached to their points are (algebraic) stacks. So, the reason that $\mathbb{A}^1_{\mathbb{C}}/(\mathbb{Z}/n\mathbb{Z})$ and $\mathfrak{h}/\mathrm{SL}_2(\mathbb{Z})$ yield spaces which don't act/look like they should, is that we should have been taking a *stack quotient*. The spaces we then produces are 'shadows' of this stack quotient in the scheme/manifold world (the replacement for stack in the complex manifold world is 'orbifold')—we took the 'best approximating scheme' to the stack, we took the 'coarse space'.

So, what does all of this have to do with torsors? Well, hopefully the above should convince you that, in general, when you have a group G acting on a space X that the 'correct' notion of quotient one can take is the 'stack quotient' denoted $[X/G]$. One may then ask natural questions such as 'is there a scheme which well-approximates the quotient stack?' In particular, you might ask when the quotient stack is representable (stacks are a fancy generalization of sheaves—they are '2-sheaves' in the sense of 2-categories—so it makes sense to ask whether a stack is representable). So, then, when is it? The answer is about what you might hope. Let $X \to Q$ be a G-equivariant map (where G acts trivially on Q). Then, $Q \cong [X/G]$ (i.e. Q represents the stack) in such a way that matches the maps $X \to Q$ and $X \to [X/G]$ if and only if $X \to Q$ is a G-torsor.

Let us summarize all of the above as follows. The notion of quotient of a group G acting on a scheme X is a sticky notion precisely because the 'correct' notion of quotient is the stack $[X/G]$—all the other notions of quotients are trying to define 'quotient schemes' which 'well-approximate' the quotient stack $[X/G]$. But, the need to approximate the 'quotient stack' by 'quotient schemes' is essentially moot when the quotient stack is actually a scheme, which is precisely the situation in which there is a G-equivariant $X \to Q$, for some scheme Q with the trivial G action, which is a torsor in which case $Q = [X/G]$.

3.4 Torsors for the Zariski and étale topologies

Now, as mentioned above, flat torsors are the most convenient setting to discuss torsors as spaces but, in practice, we want something more. Namely, we don't just want that our torsors/spaces are flat locally trivial torsors/spaces but that this happens for a more reasonable notion of cover.

We'd first like to clear up a possible confusion relating flat torsors and étale torsors. Namely, we'd like to show that elements of $\mathrm{Tors}(X_{\text{ét}}, G)$, where G is some flat algebraic X-group, are the same thing as elements of $\mathrm{Tors}(X_{\text{fl}}, G)$ that are locally trivializable for the étale topology. This will allow us to pass seamlessly between thinking about flat torsors and étale torsors.

To make this precise let's consider the following definition. Namely, for any topology $\mathcal{T}$ Sch/X coarser than the flat topology, and for any affine algebraic X-group G, let us say that a flat torsor $\mathcal{F}$ for G is *locally trivial for the $\mathcal{T}$*

topology if, in fact, one can find a covering $\{U_i \to X\}$ in $\mathcal{T}$ such that $\mathcal{F}(U_i)$ or, equivalently, $\mathcal{F}_{U_i}$ is isomorphic to the trivial torsor for all i.

We then have the following basic result:

Theorem 3.29: *Let X and G be as above. Then,* $\mathsf{Tors}(X_{\text{Ét}}, G)$ *is canonically isomorphic as pointed sets (abelian groups if G is abelian) to the subset of* $\mathsf{Tors}(X_{\text{Fl}}, G)$ *consisting of the flat torsors locally trivial for the étale topology. Similarly,* $\mathsf{Tors}(X_{\text{Zar}}, G)$ *is canonically isomorphic to the pointed subset (abelian subgroup if G is abelian) of* $\mathsf{Tors}(X_{\text{Fl}}, G)$ *consisting of those flat torsors which are locally trivial for the Zariski topology.*

Here X_{Zar} is the site with underlying category Sch/X and covers $\{U_i \to X\}$ consisting of jointly surjective collections of open embeddings (i.e. open covers are 'usual' open covers).

Proof: This is clear from definition. ■

What is much more subtle is trying to relate the torsors over the *small* sites (i.e. the small Zariski, flat, and étale sites). The reason for the extra subtlety is that it's now no longer true that the respective categories of sheaves consist of torsors on the same category. Namely, in the case of the big sites the underlying categories on which the sheaves were defined remained constant (i.e. they were all presheaves on Sch/X) whereas for the small sites the underlying category is changing. Thus, the objects of $\mathsf{Tors}(X_{\text{fl}}, G)$ and $\mathsf{Tors}(X_{\text{ét}}, G)$ are presheaves on entirely different categories, and it's not even clear how to relate the two in *either* direction. Namely, since the site $X_{\text{ét}}$ is properly contained (in the sense of both categories and covers) in X_{fl} the 'restriction map' $\mathcal{F} \mapsto \mathcal{F}|_{X_{\text{ét}}}$ from $\mathsf{Tors}(X_{\text{fl}}, G)$ to $\mathsf{Tors}(X_{\text{ét}}, G)$ is, *a priori*, not well-defined—if it were this would imply that a flat torsor is actually trivial étale locally.

Now that we have hopefully amply motivated why the small site version is different, let us actually show that under favorable conditions the two actually coincide.

The key result is the following:

Theorem 3.30: *Let G be a smooth affine X-group and $\mathcal{F}$ a G-torsor on X_{fl}. Then, $\mathcal{F}$ is automatically locally trivial for the étale topology.*

Proof: By theorem 3.24 we know that $\mathcal{F}$ is represented by an X-scheme $Y \to X$ with a G-action. Moreover, we know from Exercise 3.27 that it suffices to show that X has an étale cover $\{U_i \to X\}$ such that $Y_{U_i} \to U_i$ has a section. But, note that by the condition that $G \to X$ is smooth, and the fact that smoothness can be checked fppf locally, we know that $Y \to X$ is also smooth. But, this implies (with a small amount of work—cf. (22)) that X has an étale cover $\{U_i \to Y\}$ such that $Y_{U_i} \to U_i$ has a section, as desired. ■

So, with this we can now formualte our desired result:

Theorem 3.31: *Let X be a scheme and G a smooth affine X-group. Then, there is a canonical bijection of pointed sets (abelian groups if G is abelian)* $\mathsf{Tors}(X_{\text{ét}}, G) \cong \mathsf{Tors}(X_{\text{fl}}, G)$.

Proof (Sketch): Let us define a map $\mathsf{Tors}(X_{\text{fl}}, G) \to \mathsf{Tors}(X_{\text{ét}}, G)$ that sends a G-torsor $\mathcal{F}$ to its restriction to $X_{\text{ét}}$. Note that this is actually well-defined (in the sense that it produces a G-torsor on $X_{\text{ét}}$) by Theorem 3.30. Let us first note that this map is evidently a map of pointed sets since it sends the trivial principal G-bundle $G \to X$ to itself (since a principal G-bundle $Y \to X$, in general, gets sent to the sheaf on $X_{\text{ét}}$ represented by Y).

To see that this map is surjective, let us proceed as follows. Suppose that $\mathcal{F}$ is G-torsor on $X_{\text{ét}}$. Then, note that $\mathcal{F}$ is also representable since the sheaf $\mathcal{F}$ is étale locally representable by an affine morphism, and thus (by affine descent), actually isomorphic to a principal G-bundle $Y \to X$. Let $\mathcal{F}'$ be the sheaf on X_{fl} obtained as the representable sheaf $Y \to X$. Then, evidently $\mathcal{F}'$ is a G-sheaf on X_{fl} which is étale locally trivial (and thus flat locally trivial) so that $\mathcal{F}'$ is an element of $\mathsf{Tors}(X_{\text{fl}}, G)$. Clearly $\mathcal{F}'$ maps to $\mathcal{F}$.

Let us now show that this map is injective. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are G-torsors on X_{fl} which become isomorphic when restricted to $X_{\text{ét}}$. Note that since $\mathcal{F}_1$ and $\mathcal{F}_2$ are étale locally trivial by Theorem 3.30 these restrictions are isomorphic on a simultaneous trivialization. Thus, they give the same gluing data to a principal G-bundle $Y \to X$ and thus define the same flat torsor. ■

So, to this end, let us consider the following more general notion. Let $\mathcal{F}$ be a presheaf on Sch/X with an action of the group preheaf $\mathcal{G}$. Let us define $\mathcal{F}$ to be a *pretorsor* for $\mathcal{G}$ if for any $Y \to X$ with $\mathcal{F}(X)$ non-empty we have that

$\mathcal{G}(X)$ acts simply transitively on $\mathcal{F}(X)$. Note here that we didn't need a topology on Sch/X to make sense of this. Then, if $\mathfrak{T}$ is any Grothendieck topology on Sch/X for which $\mathcal{G}$ is a sheaf let us say that the pretorsor $\mathcal{F}$ for $\mathcal{G}$ is a $\mathfrak{T}$*-torsor* or is *locally trivial for the* $\mathfrak{T}$*-topology* if for all $Y \to X$ there exists a $\mathfrak{T}$-cover $\{U_i \to Y\}$ such that $\mathcal{F}(U_i) \neq \varnothing$ for all i.

So, for example, evidently if $\mathcal{F}$, a pretorsor for $\mathcal{G}$, is locally trivial for the $\mathfrak{T}$-topology then evidently it's locally trivial for any topology on Sch/X finer than $\mathfrak{T}$. So we can rephrase our discussion above as follows. So, if $\mathbf{Tors}(\mathscr{C}_{\mathfrak{T}}, \mathcal{G})$ denotes $\mathcal{G}$-torsors for $\mathscr{C}$ with the $\mathfrak{T}$-topology then for any two topologies mfT and $\mathfrak{T}'$ with $\mathfrak{T}$ coarser than $\mathfrak{T}'$ we get a natural inclusion $\mathbf{Tors}(\mathscr{C}_{\mathfrak{T}}, \mathcal{G}) \to \mathbf{Tors}(\mathscr{C}_{\mathfrak{T}'}, \mathcal{G})$. Let G be a flat algebraic X-group and let $\mathcal{F}$ be a pretorsor for G. Then, being a flat torsor meant that $\mathcal{F}$ was locally trivial for that flat topology. We then want to know whether or not $\mathcal{F}$ is locally trivial for a coarser, more reasonable topology than that flat topology.

The most powerful result in this direction is the following (which is essentially just a rephrasing of Theorem 3.31):

Theorem 3.32: *If G is a smooth algebraic X-group, then every flat G-torsor is locally trivial for the étale topology and every principal G-bundle $Y \to X$ is smooth. In other words, the inclusion $\mathbf{Tors}(X_{\text{Ét}}, G) \to \mathbf{Tors}(X_{\text{Fl}}, G)$ is actually an isomorphism.*

Trying to trivialize for the Zariski topology is a *much* harder feat. While it's true for for some groups (e.g. GL_n and SL_n as we'll see later on) it's generally very far from being true. In fact, Grothendieck actually classified such groups if X is a variety over k an algebraically closed field, and G is the base change to X of some algebraic group over k:

Theorem 3.33 (Grothendieck): *Let k be an algebraically closed field, X/k a variety, and G_0/k be an algebraic group. Let $D := D(G_0)$ be the derived subgroup of G_0. Then, G_0 is such that $\mathbf{Tors}(\mathrm{Open}(X), G) \to \mathbf{Tors}(X_{\mathrm{Fl}}, G)$ if and only if D is isomorphic to a product of groups of the form SL_n or Sp_{2m} for n and m non-negative.*

For example, it follows from this that no finite group has all of its torsors locally trivial for the Zariski topology. Also, groups like SO_n for $n \geqslant 3$ do not satisfy this property.

Remark 3.34: Let me end this section by making a remark about what is a very reasonable question. Namely, there is a startling symmetry to the flat torsor case which doesn't seem to extend to the étale or Zariski locally trivial case. Namely, if $Y \to X$ is a principal G-bundle (for G a flat algebraic X-group) then the cover of X needed to trivialize the torsor for the flat topology was, well, Y itself! Whereas evidently no such symmetry can happen, for example, in the étale topology (in general). Specifically, to even ask if such a thing were true for torsors locally trivial for the étale topology we'd need that $Y \to X$ was actually étale which, essentially, never happens.

Why is this? The operative thing is that we assumed that G was flat and finite presentation. If one assumed that G was étale then any principal G-bundle $Y \to X$ for the flat topology is automatically étale and self-trivializes (i.e. trivializes the torsor when pulled back along itself). As a more reasonable example, since étale algebraic X-groups are pretty rare, note that if one assumes that G is smooth then any principal G-bundle $Y \to X$ is automatically smooth, and one can self-trivalize where the covering $Y \to X$ is in the *smooth* (or *lisse*) topology.

So, it's not that the flat topology itself was making the self-trivialization happen, but more that we assumed that our groups G were flat. This happens almost always in practice (e.g. it's automatic if $X = \mathrm{Spec}(k)$. But, assuming more like G is étale is entirely too strong because it eliminates any positive-dimensional group (e.g. $\mathrm{GL}_{n,X}$, $\mathrm{SL}_{n,X}$, $\mathbf{G}_{a,X}^n, \ldots$). ♦

3.5 Torsors and cohomology

We now come to our crowning moment—to try and relate the concrete geometric spaces occuring as principal G-bundles to the not so geometric notion of (Čech) cohomology.

So, before we continue, let's make good on our promise to define non-abelian H^1—in other words, Čech cohomology when the coefficient sheaf is perhaps non-abelian. The idea will be in the same in spirit, we'll just now need to pay a little closer attention to how we take 'quotients'.

So, as before, we define it first for a cover and then pass to the limit. So, let $\mathscr{C}$ be a site, and let $\{U_i \to X\}$ be a cover of an object X of $\mathscr{C}$. Let $\mathcal{G}$ be a (possibly non-abelian) group sheaf on $\mathscr{C}$. We then define $\check{H}^1(\{U_i\}, \mathcal{G})$ as follows. Let $Z^1(\{U_i\}, \mathcal{G})$ be the set of tuples $(s_{ij}) \in \displaystyle\prod_{i,j} \mathcal{G}(U_i \times_X U_j)$ such that $s_{ij}s_{jk} = s_{ik}$ for all triples of indices

(i, j, k). We then define an equivalence relation on $Z^1(\{U_i\}, \mathcal{G})$ by declaring that $(s_{ij}) \sim (t_{ij})$ if there exists elements $g_i \in \mathcal{G}(U_i)$ for all i such that

$$s_{ij} \mid_{U_i \times_X U_j} = g_i \mid_{U_i \times X U_j} t_{ij} \mid_{U_i \times_X U_j} (g_j \mid_{U_i \times_X U_j})^{-1} \tag{25}$$

We denote the quotient set $Z^1(\{U_i\}, \mathcal{G})/\sim$ by $\check{H}^1(\{U_i\}, \mathcal{G})$ and call it the *first Čech cohomology set* of $\mathcal{G}$ on $\{U_i\}$. Notice that it's a pointed set since it has the distinguished element corresponding to the $\sim$-class of $s_{ij} = 1$ for all (i, j). We then define the *first Čech cohomology set* of $\mathcal{G}$ on X to be $\varinjlim \check{H}^1(\{U_i\}, \mathcal{G})$ as $\{U_i\}$ travels over all covers of X. Of course, if $\mathcal{G}$ is an abelian group sheaf, then this coincides with the definition we already gave.

Just as before, we have the following:

Theorem 3.35: *Let $\mathscr{C}$ be a site, and let*

$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{Q} \to 1 \tag{26}$$

be a short exact sequence of group sheaves on $\mathscr{C}$. Then, there exists an exact sequence of pointed sets

$$1 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \to \check{H}^1(T, \mathcal{K}) \to \check{H}^1(T, \mathcal{G}) \to \check{H}^1(T, \mathcal{Q}) \tag{27}$$

for all objects T of $\mathscr{C}$.

The proof is exactly the same as in the abelian case. Also, recall that an exact sequence of pointed sets means precisely that that the image of the previous map is the preimage of the distinguished element along the next map.

Remark 3.36: Beware! It's not true that an exact sequence $\{*\} \to S \to T$ of pointed sets (where $\{*\}$ denotes the set with one element) implies that $S \to T$ is injective—this differs, obviously, from the case of abelian groups where homogeneity saves the day. ♦

One can actually do better with some slight conditions on the group sheaves involved:

Theorem 3.37: *Let $\mathscr{C}$ be a site and let*

$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{Q} \to 1 \tag{28}$$

be a short exact sequence of group sheaves where $\mathcal{K}$ has image lying in $Z(\mathcal{G})$ (the center of $\mathcal{G}$). Then, one has an exact sequence of pointed sets

$$1 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \to \check{H}^1(T, \mathcal{K}) \to \check{H}^1(T, \mathcal{G}) \to \check{H}^1(T, \mathcal{Q}) \to H^2(T, \mathcal{K}) \tag{29}$$

The cohomology group $H^2(T, \mathcal{K})$ is the abelian cohomology group constructed from homological algebra.

So, let us now explain how we can relate non-abelian cohomology to torsors. In particular, let us fix our site $\mathscr{C}$ and define, for all objects T of $\mathscr{C}$, the pointed set $\mathrm{Tors}(T, \mathcal{G})$ to be $\mathrm{Tors}(\mathcal{G} \mid_T)$ (where, recall, that we defined $\mathcal{G} \mid_T$ to mean $\mathcal{G}$ restricted to the category $\mathscr{C}/T$) we now construct a map of pointed sets $\mathrm{Tors}(T, \mathcal{G}) \to \check{H}^1(T, \mathcal{G})$ as follows. For every $[\mathcal{F}] \in \mathrm{Tors}(T, \mathcal{G})$ choose a covering $\{U_i \to T\}$ such that $\mathcal{F}(U_i) \neq \varnothing$ for all i. Choose sections $\alpha_i \in \mathcal{F}(U_i)$ and note that for all (i, j) the elements $\alpha_i \mid_{U_i \times_X U_j}$ and $\alpha_j \mid_{U_i \times_X U_j}$ differ by a unique element of $\mathcal{G}(U_{ij})$: there exists a unique $s_{ij} \in \mathcal{G}(U_{ij})$ such that $\alpha_i \mid_{U_i \times_X U_j} = s_{ij}(\alpha_j \mid_{U_i \times_X U_j})$. One can then easily see that $(s_{ij}) \in Z^1(\{U_i\}, \mathcal{G})$ and that the image of this element in $\check{H}^1(T, \mathcal{G})$ is independent of all choices (the representative of $[\mathcal{F}]$ and the elements $\alpha_i \in \mathcal{F}(U_i)$). Moreover, it's evident that if $\mathcal{F}$ is the trivial torsor, then we can, for any open cover, choose the sections $\alpha_i = 1$ (the identity) so that $s_{ij} = 1$ and thus the image of the distinguished element of $\mathrm{Tors}(T, \mathcal{G})$ is the distinguished element of $\check{H}^1(T, \mathcal{G})$.

Theorem 3.38: *The map* $\mathrm{Tors}(T, \mathcal{G}) \to \check{H}^1(T, \mathcal{G})$ *is a bijection of pointed sets. Moreover, if $\mathcal{G}$ is an abelian group sheaf, it's an isomorphism of abelian groups.*

Proof (Sketch): We leave the details of this as an exercise, but since it's so simple, let us explain what the inverse is. Suppose that (s_{ij}) is an element of $\check{H}^1(T, \mathcal{G})$. Then, by definition, (s_{ij}) really belongs to $\check{H}^1(\{U_i\}, \mathcal{G})$ for some cover $\{U_i \to T\}$. Let us then build a torsor on T as follows. For each U_i consider the trivial torsor $\mathcal{G} \mid_{U_i}$. Note then that the automorphisms of $\mathcal{G} \mid_{U_i \times_X U_j}$ is precisely $\mathcal{G}(U_i \times_X U_j)$. So, on the overlaps $U_i \times_X U_j$ let us glue $\mathcal{G} \mid_{U_i}$ and $\mathcal{G} \mid_{U_j}$ together by the isomorphism $\mathcal{G} \mid_{U_i \times_X U_j} \to \mathcal{G} \mid_{U_i \times_X U_j}$ given by s_{ij}. One can check from the cocycle condition that this is a well-defined gluing data and thus we can use this to the $\mathcal{G} \mid_{U_i}$ to a $\mathcal{G}$-torsor $\mathcal{F}$. We then define the image of (s_{ij}) under $\check{H}^1(T, \mathcal{G}) \to \mathrm{Tors}(T, \mathcal{G})$ to be $[\mathcal{F}]$. ■

Exercise 3.39: *Fill in the details of the proof sketch for Theorem 3.38.*

Combining this result and Theorem 3.35 we see that for every short exact sequence of group sheaves

$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{Q} \to 1 \tag{30}$$

on $\mathscr{C}$ and for every T an object of $\mathscr{C}$ we get an exact sequence of pointed sets (or abelian groups of $\mathcal{G}$ is abelian)

$$1 \to \mathcal{K}(T) \to \mathcal{G}(T) \to \mathcal{Q}(T) \to \mathsf{Tors}(T, \mathcal{K}) \to \mathsf{Tors}(T, \mathcal{G}) \to \mathsf{Tors}(T, \mathcal{Q}) \tag{31}$$

where the maps $\mathsf{Tors}(T, \mathcal{K}) \to \mathsf{Tors}(T, \mathcal{G})$ and $\mathsf{Tors}(T, \mathcal{G}) \to \mathsf{Tors}(T, \mathcal{Q})$ are the induced torsors construction. This gives us a concrete way of understanding the obstruction to the surjectivity $\mathcal{G}(T) \to \mathcal{Q}(T)$ in terms of sheaves or, often, (in light of our discussion of flat torsors) spaces. Explain how to get connecting homomorphism via thinking about torsors.

Exercise 3.40: *Write down explicitly what the connecting homomorphism $\mathcal{Q}(T) \to \mathsf{Tors}(T, \mathcal{K})$ is in terms of torsors (Hint: think about associating to $q \in \mathcal{Q}(T)$ its fiber in $\mathcal{G}(T)$).*

4 Examples

Now that we have built up all of the theory, we should give several interesting examples of torsors

4.1 $\mathrm{GL}_{n,X}$-torsors

We start with $\mathrm{GL}_{n,X}$-torsors not because $\mathrm{GL}_{n,X}$ is the simplest group, but because $\mathrm{GL}_{n,X}$-torsors have a particularly nice form. So, let us begin by noting that since $\mathrm{GL}_{n,X}$ is a smooth algebraic X-group (and since smooth implies flat) we know from Theorem 3.24 that every $\mathrm{GL}_{n,X}$-torsors are the same thing as principal $\mathrm{GL}_{n,X}$-bundles.

So, we're looking for X-schemes $f : Y \to X$ with a $\mathrm{GL}_{n,X}$-action which are locally isomorphic to $\mathrm{GL}_{n,X}$. Now, a first step in this process might be to think about how we can even build large classes of examples of of X-schemes which carry a $\mathrm{GL}_{n,X}$-action. Well, there is a somewhat natural candidate. Namely, $\mathrm{GL}_{n,X}$ is essentially defined to be the automorphisms of an n-dimensional vector space, and so perhaps we should start the process of finding principal $\mathrm{GL}_{n,X}$-bundles by thinking about X-schemes $f : Y \to X$ whose fibers are 'vector spaces'.

Of course, we already know what such objects are—they're vector bundles! Namely, let $p : Y \to X$ be a rank n vector bundle (i.e. $Y = \mathbf{V}(\mathcal{E}) := \underline{\mathrm{Spec}}(\mathrm{Sym}(\mathcal{E}^\vee))$ for some $\mathcal{O}_X$-module $\mathcal{E}$ locally isomorphic on X to $\mathcal{O}_X^n$). Then, we have a natural action of $\mathrm{GL}_{n,X}$ on Y (as an X-scheme). But, unfortunately, these don't stand a chance of being principal $\mathrm{GL}_{n,X}$-bundles. Namely, over a geometric point $\overline{x}$ we'd have to have that the fiber $Y_{\overline{x}}$ should be isomorphic to $\mathrm{GL}_{n,\overline{x}}$, but it's not—it's isormorphic to $\mathbb{A}^n_{\overline{x}}$. These aren't isomorphic for any $n \geqslant 1$. For $n > 1$ it's for dimension reasons, and for $n = 1$ it follows by looking at units.

But, again, this is not a total failure. Our goal was just to *first* find natural spaces that $\mathrm{GL}_{n,X}$ acts on and then, hopefully, produce principal $\mathrm{GL}_{n,X}$-bundles from these. So, how can take our vector bundle $\mathbf{V}(\mathcal{E})$ and produce from it a principal $\mathrm{GL}_{n,X}$-bundle? Well, while it is *not* true that $\mathrm{GL}_{n,X}$ acts simply transitively on affine n-space (this is what caused $\mathbf{V}(\mathcal{E})$ to not be the torsor itself) it *does* act simply transitively on a natural object associated to affine n-spaces—the set of *frames*. Namely, let us define a *framing* of $\mathcal{E}$ on an X-scheme $f : T \to X$ to be an isomorphism $f^*\mathcal{E} \xrightarrow{\approx} \mathcal{O}_T^n$. Then, intuituively, the framings of $\mathcal{E}$ on T are the sets of ordered bases of $\mathcal{E}$—where, we can interpret this set of bases as being empty.

So, let us define the *frame bundle* associated to $\mathcal{E}$ to be the following sheaf $\mathrm{Frame}_{\mathcal{E}} : \mathsf{Sch}/X \to \mathsf{Set}$:

$$\mathrm{Frame}_{\mathcal{E}}(T) := \left\{ f \in \mathrm{Hom}_{\mathcal{O}_T}(f^*\mathcal{E}, \mathcal{O}_T^n) : f \text{ an isomorphism} \right\} \tag{32}$$

in other words (for those familiar with stack-y math) $\mathrm{Frame}_{\mathcal{E}}$ is the sheaf $\mathrm{Isom}(\mathcal{E}, \mathcal{O}_X^n)$ for the stack Qcoh on Sch/X (with the fppf topology). The fact that $\mathrm{Frame}_{\mathcal{E}}$ is a sheaf on X_{Fl} follows from so-called *fppf descent for quasi-coherent sheaves* (e.g. see this). Note that $\mathrm{GL}_{n,X}$ naturally acts on $\mathrm{Frame}_{\mathcal{E}}$ by post-composition.

Exercise 4.1: *Show that if $\mathcal{E} = \mathcal{L}$ is a line bundle then* $\mathrm{Frame}_{\mathcal{L}}$ *is just* $\mathbf{V}(\mathcal{L}) - \{\mathbf{0}\}$ *(i.e. the complement of the zero section for the space associated to the line bundle).*

We claim that $\mathrm{Frame}_{\mathcal{E}}$ is actually a principal $\mathrm{GL}_{n,X}$-bundle locally trivial for the Zariski topology. But, this is fairly clear. Namely, let us note that if $\mathrm{Frame}_{\mathcal{E}}(T) \neq \varnothing$ then $\mathrm{Frame}_{\mathcal{E}}(T) = \mathrm{Aut}(\mathcal{O}_T^n)$ with the left action of $\mathrm{GL}_{n,T}$, but $\mathrm{Aut}(\mathcal{O}_T^n) = \mathrm{GL}_{n,T}$ and so the simply transitivity follows. Moreover, to see that $\mathrm{Frame}_{\mathcal{E}}$ is Zariski locally trivial we note that X has an open cover $\{U_i\}$ where $\mathcal{E}\mid_{U_i} \cong \mathcal{O}_{U_i}^n$ so that $\mathrm{Frame}_{\mathcal{E}}(U_i) \neq \varnothing$ for all i.

The converse is also true:

Theorem 4.2: *The association $\mathcal{E} \mapsto \mathrm{Frame}_{\mathcal{E}}$ is a bijection of pointed sets*

$$\left\{\begin{matrix}\text{Rank } n \text{ vector}\\ \text{bundles on } X\end{matrix}\right\}/\text{iso.} \xrightarrow{\approx} \mathrm{Tors}(X_{\mathrm{Fl}}, \mathrm{GL}_{n,X}) \tag{33}$$

where the left hand side has distinguished element the isomorphism class of $\mathcal{O}_X^n$.

Proof (Sketch): One proves that for any 'ringed site' $(\mathscr{C}, \mathcal{O}_{\mathscr{C}})$ (i.e. a site with a fixed sheaf of rings) the Čech cohomology group $\check{H}^1(\mathscr{C}, \mathrm{GL}_n(\mathcal{O}_{\mathscr{C}}))$ classifies rank n locally free $\mathcal{O}_{\mathscr{C}}$-modules—this is standard gluing. Thus, using Theorem 3.38 we know that $\mathrm{Tors}(X_{\mathrm{Fl}}, \mathrm{GL}_{n,X})$, which is equal to $\check{H}^1(X_{\mathrm{Fl}}, \mathrm{GL}_n(\mathcal{O}_{X_{\mathrm{Fl}}}))$ classifies rank n locally free $\mathcal{O}_{X_{\mathrm{Fl}}}$-modules. One then uses fppf descent to show that every such $\mathcal{O}_{X_{\mathrm{Fl}}}$-module is the pullback to X_{Fl} of a vector bundle on X which, proves the result.

An explicit inverse takes $\mathcal{F}$ to $\mathcal{F} \times_{\mathrm{GL}_{n,X}} \mathcal{O}_X^n$—this is the contracted product (which one can make sense of in this context with the exact same formula we laid out for the contracted product of torsors). ■

To summarize the above sketch one shows that in vast generality $\check{H}^1$ of GL_n classifies vector bundles. The real key step is that every 'flat vector bundle' is Zariski locally trivial, which required the theory of fppf descent (whose relevant statement here is that $\mathrm{Qcoh}(X_{\mathrm{Fl}}) = \mathrm{Qcoh}(\mathrm{Open}(X))$). Note that this is not at all at odds with Theorem 3.33 since $D(\mathrm{GL}_n) = \mathrm{SL}_n$.

Now, note that the only $n \geqslant 1$ for which $\mathrm{GL}_{n,X}$ is abelian is $\mathbf{G}_{m,X}$. There we can make the following more precise statement of Theorem 4.2:

Theorem 4.3: *The association $\mathcal{L} \mapsto \mathrm{Frame}_{\mathcal{L}}$ is an isomorphism of abelian groups* $\mathrm{Pic}(X) \xrightarrow{\approx} \mathrm{Tors}(X_{\mathrm{Fl}}, \mathbf{G}_{m,X})$.

Remark 4.4: Note in particular that associated to a line bundle $\mathcal{L}$ is a $\mathbf{G}_{m,X}$-torsor. People often times say "line bundles are $\mathbf{G}_{m,X}$-torsors". This confused to me to no end since I thought this mean that there was some simply transitive action of $\mathbf{G}_{m,X}$ on the sheaf $\mathcal{L}$ making it into a $\mathbf{G}_{m,X}$-torsor. Of course, people are just being imprecise: line bundles are not <u>equal</u> to $\mathbf{G}_{m,X}$-torsors, just naturally in bijection with. ♦

4.2 $\mathrm{SL}_{n,X}$-torsors

Let us now try and understand $\mathrm{SL}_{n,X}$ torsors. Note that since $\mathrm{SL}_{n,X}$ is perfect (i.e. $D(\mathrm{SL}_{n,X}) = \mathrm{SL}_{n,X}$) it follows from Theorem 3.33 that every principal $\mathrm{SL}_{n,X}$-bundle should Zariski locally trivial. But, how do we figure out precisely what they are?

Well, the obvious idea is to try and utilize our understanding of Theorem 3.35 to try and understand how the torsors of $\mathrm{SL}_{n,X}$ are put together from the torsors of $\mathrm{GL}_{n,X}$ and $\mathbf{G}_{m,X}$ since we have the defining exact sequence

$$1 \to \mathrm{SL}_{n,X} \to \mathrm{GL}_{n,X} \to \mathbf{G}_{m,X} \to 1 \tag{34}$$

where $\mathrm{GL}_{n,X} \to \mathbf{G}_{m,X}$ is the determinant map. Indeed, considering the sequence (27) we obtain the exact sequence of pointed sets

$$1 \to \mathrm{SL}_n(\mathcal{O}_T(T)) \to \mathrm{GL}_n(\mathcal{O}_T(T)) \to \mathcal{O}_T(T)^\times \to \check{H}^1(T, \mathrm{SL}_{n,X}) \to \check{H}^1(T, \mathrm{GL}_{n,X}) \to \check{H}^1(T, \mathbf{G}_{m,X}) \tag{35}$$

which should be useful in understanding $\check{H}^1(T, \mathrm{SL}_{n,X})$ since we know that $\check{H}^1(\mathrm{GL}_{n,X})$ is rank n vector bundles on T and $\check{H}^1(T, \mathrm{GL}\, m, X)$ are line bundles on T.

Note that the map $\mathrm{GL}_n(\mathcal{O}_T(T) \to \mathcal{O}_T(T)^\times$ in (35) is actually surjective. Indeed, for any $\alpha \in \mathcal{O}_T(T)^\times$ the matrix with α in the top left corner, 1's along the rest of the diagonal, and zeros otherwhere is in $\mathrm{GL}_n(\mathcal{O}_T(T))$ and has determinant α. Thus, we have an exact sequence

$$1 \to \check{H}^1(T, \mathrm{SL}_{n,X}) \to \check{H}^1(T, \mathrm{GL}_{n,X}) \to \check{H}^1(T, \mathbf{G}_{m,X}) \tag{36}$$

which gives a very strong indication of what $\check{H}^1(T, \mathrm{SL}_{n,X})$ is. This follows because the map $\check{H}^1(T, \mathrm{GL}_{n,X}) \to \check{H}^1(T, \mathbf{G}_{m,X})$ sends $\mathrm{Frame}_{\mathcal{E}}$ to $\mathrm{Frame}_{\det(\mathcal{E})}$ where $\det(\mathcal{E})$ is the determinant bundle $\wedge^n \mathcal{E}$.

Exercise 4.5: *Prove this claim.*

Thus, we see that the image of $\check{H}^1(T, \mathrm{SL}_{n,X}) \to \check{H}^1(T, \mathrm{GL}_{n,X})$ is precisely those $\mathrm{Frame}_{\mathcal{E}}$ with $\det(\mathcal{E}) = \mathcal{O}_X$. In other worrds, it corresponds to rank n vector bundles with trivial determinant bundle.

The slightly non-obvious result is that the map $\check{H}^1(T, \mathrm{SL}_{n,X}) \to \check{H}^1(T, \mathrm{GL}_{n,X})$ is injective (recall Remark 3.36) so that we have the following:

Theorem 4.6: *The map* $\mathcal{E} \mapsto \mathrm{Frame}_{\mathcal{E}}$ *is a bijection of pointed sets*

$$\left\{ \begin{array}{c} \text{Rank } n \text{ vector} \\ \text{bundles } \mathcal{E} \text{ on } X \\ \text{with } \det(\mathcal{E}) \cong \mathcal{O}_X \end{array} \right\} /\text{iso.} \xrightarrow{\approx} \mathrm{Tors}(X_{\mathrm{Fl}}, \mathrm{SL}_{n,X}) \tag{37}$$

4.3 $\underline{G}$-torsors

We now get to what is, perhaps, the most important examples of torsors at least insofar as étale cohmology/étale fundamental groups are concerned. The idea rougly being that if $Y \to X$ is a torsor for a constant finite group, then this should mean that $Y \to X$ is a finite Galois cover with Galois group G.

Before we get into it, let's first clarify what we mean by $\underline{G}$. Namely, let G be a finite group in the classical sense of the term. We can then create that *constant group scheme* $\underline{G}$ over X (perhaps should be denoted as $\underline{G}_X$, but hopefully it will be clear from context) as follows. The underlying scheme of $\underline{G}$ is $\displaystyle\bigsqcup_{g \in G} X$—we have a disjoint union of a number of copies of X indexed by the elements of G. We then define a multiplication map $m : \underline{G} \times_X \underline{G} \to \underline{G}$ by noting that

$$\underline{G} \times_X \underline{G} = \bigsqcup_{(g,h) \in G^2} X \tag{38}$$

and having m send the (g,h)-copy of X in $\underline{G} \times_X \underline{G}$ by the identity to the gh-copy of X in $\underline{G}$. One can check that this is, indeed, a group scheme over X which is a flat algebraic (in fact, finite) X-group.

For us, probably the most common example of this will be the constant group scheme $\underline{\mathbb{Z}/n\mathbb{Z}}$ since this is the one which pops up most prominently in étale cohomology.

Exercise 4.7: *Over $\mathbb{C}$ we often times identify the set of n^{th}-roots of unity (non-canonically) with $\mathbb{Z}/n\mathbb{Z}$ by picking a generator—by picking a primitive n^{th}-root of unity. Find necessary and sufficient conditions on X such that the* sheaves *$\mu_{n,X}$ and $\underline{\mathbb{Z}/n\mathbb{Z}}_X$ are isomorphic.*

So, now, the key result to understanding $\underline{G}$ torsors is the following:

Theorem 4.8: *Let X be a connected scheme and let $f : Y \to X$ be a finite Galois cover with Galois group G. Then, $f : Y \to X$ is a principal $\underline{G}$-bundle.*

In case we've forgotten, recall that a *finite Galois cover* is a finite étale surjection $Y \to X$ with Y connected and such that $= \mathrm{Aut}(Y/X)$ acts transitively on the geometric points of Y lying over any geometric point of X.

Also, before we begin the proof of Theorem 4.8 it helps to clear up a possibly confusing point. Namely, if $f : Y \to X$ is a finite Galois cover with $G := \mathrm{Aut}(Y/X)$. Then, evidently the abstract group G acts on Y as an X-scheme, but Theorem 4.8 implies that Y has an action of $\underline{G}$. The connection is as follows:

Exercise 4.9: *Let G be an abstract group. Show that an action of G on an X-scheme Y is the same thing as an action of the group scheme $\underline{G}_X$ on the X-scheme Y.*

Proof (Theorem 4.8): To prove this result it suffices to show that $Y \times_X Y \cong \underline{G} \times_X Y$ in a $\underline{G}$-equivariant manner. Now, note that $Y \times_X Y \to Y$ is a finite étale cover but which might not be Galois since $Y \times_X Y$ may be disconnected. But, if we decompose $Y \times_X Y = \displaystyle\bigsqcup_i Z_i$ where the Z_i are the connected components of $Y \times_X Y$, then each $Z_i \to X$ is evidently a finite Galois cover.

Now, note that the diagonal map $\Delta : Y \to Y \times_X Y$ is a section of the map $Y \times_X Y \to Y$ and thus, since $Y \times_X Y \to Y$ is finite étale, is a clopen embedding. Thus, Δ is an isomorphism of Y onto one of the components Z_i, call it Z_0. Consider now that if we postcompose Δ with the automorphism $g \in G$ of $Y \times_X Y$, obtained by having G act on the first factor, we obtain a new clopen embedding $g \circ \Delta$. Note though that this picks out a different component of Z_{i_g} of $Y \times_X Y$ since it contains a different geometric point of $Y \times_X Y$ than that of Z_0. Thus, we see that we obtain for every $g \in G$ a component Z_{i_g} of $Y \times_X Y$ corresponding to the clopen embedding $g \circ \Delta$.

But, note that this must account for all of the components Z_i since G acts transitively on the geometric fibers of $Y \to X$. Thus, we obtain a decomposition

$$Y \times_X Y \cong \bigsqcup_{g \in G} Z_{i_g} \cong \underline{G} \times_X Y \tag{39}$$

as Y-schemes. Moreover, this is actually a $\underline{G}$-equivariant isomorphism since, by construction, $\underline{G}$ acts on $Y \times_X Y$ by the action on the Z_{i_g} that comes from permuting the indices, which is precisely the action on $\underline{G} \times_X Y$. ■

So, from this we see that every finite Galois cover with group $\underline{G}$ gives us an $\underline{G}$-torsor. But, note, we certainly cannot get all such torsors in this way. Why? Well, we assumed that Y was connected and even for the trivial torsor this fails to be true! So, we somehow want to take any principal $\underline{G}$-bundle and break it up into connected principal $\underline{G}$-subbundles.

That said, we do have the following partial converse to Theorem 4.8

Theorem 4.10: *Let G be a finite group, X a connected scheme, and $f : Y \to X$ a principal $\underline{G}$-bundle with Y connected. Then, Y is a finite Galois cover with automorphism group G.*

Proof: Note that since $\underline{G}$ acts on Y as an X-scheme we obtain a homomorphism $G \to \mathrm{Aut}(Y/X)$ which is necessarily injective since distinct elements of G act distinctly on $\underline{G}$ and Y is étale locally $\underline{G}$. Note though that $\deg(f) = |G|$ since degree can be checked étale locally, and we know that étale locally an $\underline{G}$-torsor (since it's smooth) is isomorphic to $\underline{G}$ which has degree $|G|$. Thus, we see that $\deg(f) = |G| \leqslant |\mathrm{Aut}(Y/X)|$.

Now, note that f is necessarily finite étale surjective. Indeed, this is true for $\underline{G}$, these properties can be checked étale locally, and Y is étale locally $\underline{G}$. Thus, f is a finite étale surjection with $\deg(f) \leqslant |\mathrm{Aut}(Y/X)|$. But, $|\mathrm{Aut}(Y/X)| \leqslant \deg(f)$ always, and thus $\deg(f) = |\mathrm{Aut}(Y/X)|$. This implies by basic theory that $Y \to X$ is a Galois cover, and by degree considerations $G \cong \mathrm{Aut}(Y/X)$ as desired. ■

So, where are these other $\underline{G}$-torsors coming from? Well, note that if $H \subsetneq G$ is a proper subgroup then any connected finite étale cover $f : Y \to X$ with Galois group H gives rise to a $\underline{G}$-torsor by looking at the induced G-torsor $\varphi_*(Y)$ under the inclusion $\varphi : \underline{H} \to \underline{G}$. Thus, all we need to produce a principal $\underline{G}$-torsor is a finite Galois cover of Y with automorphism group $H \subseteq G$.

Now, let's fix a geometric point $x \in X$ nd, note that we obtain such things precisely by considering homomorphisms $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X, x), G)$. Namely, such a a homomorphism ρ gives us a connected pointed Galois cover $(Y, y) \to (X, x)$ with Galois group $\rho(\pi_1^{\text{ét}}(X, x))$ which then, by induction from the inclusion $\rho(\pi_1^{\text{ét}}(X, x)) \hookrightarrow G$, gives us a principal $\underline{G}$-bundle. Note though that Theorem 4.8 and Theorem 4.10 had no dependence on the choice of a geometric base point, and so we should eliminate this dependency. It follows from the basic theory of étale fundamental groups that different choices of base points differ by inner automorphisms and such homomorphisms will differ by inner automorphisms of G.

Thus, we can put all of this together to obtain the following theorem which we've partially proved, (hopefully) amply motivated, and the details of which we leave as an exercise:

Theorem 4.11: *Let X be a connected scheme and x a geometric point of X. Suppose in addition that G is a finite abstract group. Define a map*

$$\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X, x), G)/\mathrm{Inn}(G) \to \mathsf{Tors}(X_{\mathrm{Fl}}, \underline{G}) \tag{40}$$

by sending a homomorphism $\rho : \pi_1^{\text{ét}}(X, x) \to G$ to the principal $\underline{G}$-bundle $\varphi_*(Y)$ where Y is the principal $\underline{\rho(\pi_1^{\text{ét}}(X, x))}$-bundle obtained from Theorem 4.8 and φ is the inclusion $\underline{\rho(\pi_1^{\text{ét}}(X, x)} \hookrightarrow \underline{G}$. Then, the map (40) is a bijection of pointed sets where the trivial homomorphisms (which is the only element of its $\mathrm{Inn}(G)$-orbit) is the distinguished element of the left hand side.

Now, if we assume that G is abelian then $\mathrm{Inn}(G)$ is trivial, and thus the left hand side of (40) reduces to $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X, x))$ which is an abelian group. We then obtain the following:

Corollary 4.12: *Let G be a finite abelian group, X a connected scheme, and x a geometric point of X. Then, the map from Theorem 4.11 induces an isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X,x),G) \xrightarrow{\approx} \mathsf{Tors}(X_{\mathrm{Fl}},G) \tag{41}$$

Let us give a final note that, evidently, $\mathrm{Aut}(G)$ acts on $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X,x),G)$ on the right, and if we consider the quotient $\mathrm{Hom}_{\mathrm{cont.}}(\pi_1^{\text{ét}}(X,x),G)/\mathrm{Aut}(G)$ we get the pointed set of all connected finite Galois covers of X with Galois group isomorphism to a subgroup of G.

4.4 Torsors over quasi-coherents

We now discuss a baby case of classifying 'torsors over quasi-coherents' on X_{Fl}. We have alluded to this notion several times throughout this note, so let us briefly give a rigorous description of what this means. Namely, denote by $\mathcal{O}_{X_{\mathrm{Fl}}}$ the sheaf on X_{Fl} given by sending an X-scheme T to $\mathcal{O}_T(T)$. Note that we have also called this the *additive group* and denoted $\mathbf{G}_{a,X}$—this makes sense since $\mathcal{O}_{X_{\mathrm{Fl}}}$ is representable by $\mathbb{A}^1_X$. We then define an abelian sheaf $\mathcal{F}$ on X_{Fl} to be an $\mathcal{O}_{X_{\mathrm{Fl}}}$*-module* if, as one would expect, for all T $\mathcal{F}(T)$ is a $\mathcal{O}_{X_{\mathrm{Fl}}}(T)$-module such that for all maps $S \to T$ of X-schemes the map $\mathcal{F}(T) \to \mathcal{F}(S)$ is a map of abelian groups intertwining the module operations under the ring map $\mathcal{O}_{X_{\mathrm{Fl}}}(T) \to \mathcal{O}_{X_{\mathrm{Fl}}}(S)$. Finally, let us say that $\mathcal{F}$ is *quasi-coherent* if locally (for the flat topology) we can write $\mathcal{F}$ as a quotient of some (possibly infinite) direct sum of $\mathcal{O}_{X_{\mathrm{Fl}}}$.

Now, the big result about quasi-coherent sheaves on X_{Fl} is that they are all induced by Zariski quasi-coherents. Namely, suppose that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module in the classical sense (so that it's a sheaf on $\mathrm{Open}(X)$). Then, we define a presheaf $\mathcal{F}_{\mathrm{Fl}}$ on X_{Fl} by definining $\mathcal{F}_{\mathrm{Fl}}(T) := (f^*\mathcal{F})(T)$ if the structure map $T \to X$ is denoted by f.

We then have the following:

Theorem 4.13 (Quasi-coherent descent): *Let X be a scheme. Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the presheaf $\mathcal{F}_{\mathrm{Fl}}$ is a quasi-coherent $\mathcal{O}_{X_{\mathrm{Fl}}}$-module. Moreover, the functor*

$$\mathsf{Qcoh}(\mathcal{O}_X) \to \mathsf{Qcoh}(\mathcal{O}_{X_{\mathrm{Fl}}}) : \mathcal{F} \mapsto \mathcal{F}_{\mathrm{Fl}} \tag{42}$$

is an equivalence of categories.

This then makes it seem plausible that quasi-coherent torsors (i.e. torsors under an abelian group sheaf $\mathcal{G}$ that is quasi-coherent) should all be Zariski locally trivial. This is, in fact, the case:

Theorem 4.14: *Let X be a scheme and $\mathcal{G}_{\mathrm{Fl}}$ a quasi-coherent $\mathcal{O}_{X_{\mathrm{Fl}}}$-module with $\mathcal{G}$ a quasi-coherent $\mathcal{O}_X$-module. Then, the obvious map*

$$H^1(\mathrm{Open}(X),\mathcal{G}) \to H^1(X_{\mathrm{Fl}},\mathcal{G}_{\mathrm{Fl}}) \tag{43}$$

is an isomorphism. In particular, the obvious map

$$\mathsf{Tors}(\mathrm{Open}(X),\mathcal{G}) \to \mathsf{Tors}(X_{\mathrm{Fl}},\mathcal{G}_{\mathrm{Fl}}) \tag{44}$$

is a bijection of pointed sets. In fact, the even stronger claim that the obvious map of categories

$$\mathbf{Tors}(\mathrm{Open}(X),\mathcal{G}) \to \mathbf{Tors}(X_{\mathrm{Fl}},\mathcal{G}_{\mathrm{Fl}}) \tag{45}$$

is an equivalence.

Exercise 4.15: *Explain what the 'obvious maps' in Theorem 4.14 are.*

In particular, we obtain the following somewhat humorous corollary:

Corollary 4.16 (Serre): *Let X be a scheme. Then, every torsor over a quasi-coherent sheaf on X_{Fl} is (isomorphic to) the trivial to such torsor if and only if X is affine.*

This, in particular, tells you that what torsors over quasi-coherent modules are is somewhat more sophisticated a notion than torsors over other group sheaves since they detect affineness—and affineness is not a purely 'geometric' notion, it is of an 'analytic' or 'algebraic' nature.

Let us consider a concrete example of a torsor over a quasi-coherent. Namely, let's write down a non-trivial $\mathcal{O}_{X_{\mathrm{Fl}}}$-torsor for $X = \mathbb{A}^2_k - \{0\}$ for some field k and with $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$. Specifically, let's write down a principal $\mathcal{O}_{X_{\mathrm{Fl}}}$-bundle over $\mathbb{A}^2_k - \{0\}$. Namely, let $U_0 := D(x) \times \mathbb{A}^1_k$ and $U_1 = D(y) \times \mathbb{A}^1_k$ and note that the projection maps $U_0 \to D(x)$ and $U_1 \to D(y)$. We want to glue these two maps together to obtain a map $Y \to \mathbb{A}^2_k - \{0\}$. We do this by gluing them along $D(x) \cap D(y) = D(xy)$ by the automorphism of $\mathbb{A}^1_{D(xy)} \to \mathbb{A}^1_{D(xy)}$ obtained from (additive) translation by $\dfrac{1}{xy}$.

Exercise 4.17: *Show that $Y \to \mathbb{A}^2_k - \{0\}$ defined above is a non-trivial $\mathbf{G}_{a,X}$-torsor. Conclude that $\mathbb{A}^2_k - \{0\}$ is not affine. Why does this not also show that $\mathbb{A}^1_k - \{0\}$ is not affine?*

5 The theory of twists

5.1 Motivation

This section will, necessarily, require more background than the other sections but is also, perhaps, the most satisfying example of torsors that we have. The idea roughly is as follows (which we will try and make more precise soon). Suppose that one has a site $\mathscr{C}$ such that for all objects X of $\mathscr{C}$ one has a notion of 'X-object'. Then, a common question one might try to answer is whether one can classify all X-objects $\mathcal{F}$ which become isomorphic to a fixed X-object $\mathcal{F}_0$ 'locally'. Less cryptically, let us (in this informal setting) say that $\mathcal{F}$ is a *twist* (or *form*) of $\mathcal{F}_0$ if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}_{U_i} \cong \mathcal{F}_0 \mid_{U_i}$ as U_i-objects for all U_i.

Examples of twists are nearly ubiquitous in algebraic geometry, but let me single out a few examples that are perhaps already familiar to the reader:

1. Line bundles. Indeed, by definition, line bundles $\mathcal{L}$ are just twists of $\mathcal{O}_X$—quasi-coherent sheaves on X (quasi-coherents being the relevant notion of 'X-object' here) which are locally on X isomorphic to $\mathcal{O}_X$. Of course, more generally, vector bundles of rank n are twists of $\mathcal{O}_X^n$.

2. Quadratic twists of elliptic curves. Recall that, for example, if $E/\mathbb{Q}$ is an elliptic curve with Weierstrass equation $y^2 = f(x)$ then one defines a *quadratic twist* of E by d, denoted E_d, to be an elliptic curve of the form $dy^2 = f(x)$ where $d \in \mathbb{Q}^\times$ is a non-square. This is actually a notion of twist for the *étale topology* on $\mathrm{Spec}(\mathbb{Q})$. Namely, if our notion of 'X-object' is 'elliptic curve' then we see that E_d and E are non-isomorphic but $(E_d)_{\mathbb{Q}(\sqrt{d})} \cong E_{\mathbb{Q}(\sqrt{d})}$ so that E_d and E become isomorphic over the *étale cover* $\mathrm{Spec}(\mathbb{Q}(\sqrt{d})) \to \mathrm{Spec}(\mathbb{Q})$.

3. Genus 0 curves. Indeed, suppose that C/k is a (smooth projective geometrically integral) curve of genus 0. This, on the surface, doesn't look like a twist. But, note that if $k = \overline{k}$ then $C \cong \mathbb{P}^1_k$ (indeed Riemann-Roch implies that there is a non-constant global section of $\mathcal{O}(p)$ for any $p \in C$ which defines a degree 1 map $C \to \mathbb{P}^1_k$). Thus, if k is arbitrary then while C might not be isomorphic to $\mathbb{P}^1_k$ we have that $C_{\overline{k}} \cong \mathbb{P}^1_{\overline{k}}$ and thus, for some finite extension L/k, we must have that $C_L \cong \mathbb{P}^1_L$. Thus, if 'X-object' is the notion 'smooth projective geometrically integral curve' then the genus 0 curves are precisely the twists of $\mathbb{P}^1_k$ trivialized by some fppf cover $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ with L/k finite (really a better phrasing would be that all genus 0 curves are twist of $\mathbb{P}^1_k$ for the fpqc cover $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$).

Now, while there are many reasons that one would be interested in twists (for example, in 1. above vector bundles are just interesting!), examples 2. and 3. underlie a common method of classification of objects with twists being a pivotal component. Namely, suppose for example that you are trying to classify some type of object over a field k. One might do this in two steps. First, classify objects over $\overline{k}$ (this might be easier for the usual reasons that $\overline{k}$ is easier than k). Second, classify twists of objects from $\mathrm{Spec}(k)$ to $\mathrm{Spec}(\overline{k})$. This is how an incredible number of ostensibly untenable classification problems (e.g. in the theory of group schemes) can be done.

OK, great. Twists sound awesome. But what is the relevancy to torsors? Well, it's actually fairly simple. Namely, suppose that $\mathcal{F}$ is an X-object and let $\mathrm{Aut}(\mathcal{F})$ be its automorphism sheaf: for every map $T \to X$ let $\mathrm{Aut}(\mathcal{F})(T) :=$

$\mathrm{Aut}(\mathcal{F}_T)$. Then, the following motto is the relationship between torsors and twists: twists of $\mathcal{F}$ are precisely $\mathrm{Aut}(\mathcal{F})$-torsors. The reason that this is a motto and not a theorem is purely a matter of preciseness. Namely, it requires a non-trivial amount of setup to explain what the right context one can make the above rigorous is (the short answer: stacks).

Note that it is somewhat perverse to think that twists are a special case of torsors when, in fact, the opposite is also true! Namely, we essentially defined a $\mathcal{G}$-torsor to be a $\mathcal{G}$-sheaf locally isomorphic to $\mathcal{G}$ (the trivial torsors). This is not at all at odds with the previous paragraph since the automorphism sheaf of $\mathcal{G}$ (the trivial torsors) as a $\mathcal{G}$-sheaf is, well, $\mathcal{G}$ (or $\mathcal{G}^{\mathrm{op}}$). Thus, the previous paragraph reduces in this case to the deep statement '$\mathcal{G}$-torsors are $\mathcal{G}$-torsors'.

5.2 Rigorous statements

This part will require significantly more background than the previous parts. If the reader is not familiar with the the theory of stacks, they should just skip this part and read the examples of the next section taking for granted that 'everything works'. If you want to read this section and haven't seen stacks before, just think of them as 'sheaves of categories where one can glue'.

So, let us setup some notation. Let $\mathscr{C}$ be a site and $\mathscr{S}$ a stack over $\mathscr{C}$. For any object X of $\mathscr{C}$ and any object $\mathcal{F}_0$ of the fiber $\mathscr{S}(X)$ let us define the pointed set of *twists*, denoted $\mathsf{Twist}(\mathcal{F}_0)$, to be the isomorphism classses $\mathcal{F}$ of objects of $\mathscr{S}(X)$ such that for some cover $\{U_i \to X\}$ one has that $\mathcal{F}_{U_i} \cong (\mathcal{F}_0)U_i$ as objects of $\mathscr{S}(U_i)$. The distinguished element of this set obviously being the isomorphism class of $\mathcal{F}_0$. For a given fixed open cover $\{U_i \to X\}$ let us denote by $\mathsf{Twist}(\{U_i\}, \mathcal{F}_0)$ the pointed set consisting of those isomorphism classes in $\mathsf{Twist}(\{U_i\}, \mathcal{F}_0)$ that become isomorphic to $\mathcal{F}_0$ on the cover $\{U_i\}$. Note that if $\{V_j\}$ is a refinement of $\{U_i\}$ then we get an obvious map of pointed sets $\mathsf{Twist}(\{U_i\}, \mathcal{F}_0) \to \mathsf{Twist}(\{V_j\}, \mathcal{F}_0)$ and so clearly

$$\mathsf{Twist}(\mathcal{F}_0) = \varinjlim \mathsf{Twist}(\{U_i\}, \mathcal{F}_0) \tag{46}$$

as the colimit ranges over covers of X.

With all of this setup, we can now state the main theorem concerning the relationship between twists and torsors:

Theorem 5.1: *Let $\mathscr{C}$ be a site and $\mathscr{S}$ a stack over $\mathscr{C}$. Then, for any object X of $\mathscr{C}$ and any cover $\{U_i \to X\}$ there is a canonical bijection of pointed sets*

$$\check{H}^1(\{U_i\}, \mathrm{Aut}(\mathcal{F}_0)) \xrightarrow{\approx} \mathsf{Twist}(\{U_i\}, \mathcal{F}_0) \tag{47}$$

which, upon passing to the limit, gives a canonical bijection of pointed setse

$$\check{H}^1(X, \mathrm{Aut}(\mathcal{F}_0)) \xrightarrow{\approx} \mathsf{Twist}(\mathcal{F}_0) \tag{48}$$

In particular, we have canonical bijections of pointed sets

$$\mathsf{Tors}(\mathrm{Aut}(\mathcal{F}_0)) \xrightarrow{\approx} \mathsf{Twist}(\mathcal{F}_0) \tag{49}$$

Proof (Sketch): The claimed bijection in (49) follows from that of (48) from Theorem 3.38. Moreover, by the discussion preceeding this theorem we see that the claimed bijection (48) will follow from (47). Finally, (47) is true since one can check that an element of the left hand side of (47) is precisely data to glue the objects $(\mathcal{F}_0)_{U_i}$ over U_i together to an object over X. ∎

Even though it was made to sound in the proof of Theorem 5.1 that the main bijection was (48) and that the bijection in (49) was derived from it, it's easy to make this final bijection explicit (and simple!). Namely, suppose that $\mathcal{F}$ is a twist of $\mathcal{F}_0$. Then, the Isom-sheaf $\mathrm{Isom}(\mathcal{F}, \mathcal{F}_0)$ is easily seen to be an $\mathrm{Aut}(\mathcal{F}_0)$-torsor and the map in (49) is just that: sending $\mathcal{F}$ to $\mathrm{Isom}(\mathcal{F}, \mathcal{F}_0)$.

This clarifies, for example, the discussion in §4.1 since $\mathrm{GL}_{n,X} = \mathrm{Aut}(\mathcal{O}_X^n)$ (where we're viewing the objects as elements of the stack Qcoh over X_{Fl}) and thus $\mathrm{GL}_{n,X}$-torsors twists of $\mathcal{O}_X^n$ or, equivalently, rank n vector bundles. In fact, a minimally close inspection of the proof of Theorem 4.2 shows that it's the same as the above discussion since the frame bundle $\mathrm{Frame}_{\mathcal{E}}$ is just the Isom-sheaf $\mathrm{Isom}(\mathcal{E}, \mathcal{O}_X^n)$.

5.3 Examples

We now put Theorem 5.1 to great effect by considering a few specific 'general' examples, and then using them to prove very specific classification results.

Affine schemes Recall that the fibered category Aff over X_{Fl} with fiber over T of affine T-schemes (with morphisms isomorphisms) is a stack. Thus, we obtain the following result:

Theorem 5.2: *Let X be a scheme, and T an X-scheme. Then, for any affine T-scheme $f : Y \to X$ the set* $\mathrm{Twist}(Y_0)$ *is isomorphic to $\check{H}^1(T, \mathrm{Aut}(Y_0))$ as pointed sets. More specifically, for any flat cover $\{U_i \to T\}$ the isomorphism classes of affine T-schemes Y such that $Y_{U_i} \cong (Y_0)_{U_i}$ for all i is isomorphic, as pointed sets, to $\check{H}^1(\{U_i\}, \mathrm{Aut}(Y_0))$.*

Remark 5.3: Note that X in Theorem 5.2 really played no role. One might as well assume that $X = \mathrm{Spec}(\mathbb{Z})$. ♦

Note that, in fact, one doesn't need to add the adjective 'affine' to Y. Any twist of Y_0 as a scheme (i.e. a scheme locally isomorphic to Y_0 in the fppf topology) is (by affine descent) already affine.

As an example of this, let us show that if k is any field and Y/k is any scheme such that $Y_{k^{\mathrm{sep}}} \cong \mathbb{A}^1_{k^{\mathrm{sep}}}$ then $Y = \mathbb{A}^1_k$. Indeed, by standard arguments it suffices to show that if L/k is a finite separable extension and $Y_L \cong \mathbb{A}^1_L$ then $Y \cong \mathbb{A}^1_k$. But, note that this means that Y is an element of $\mathrm{Twist}(\{\mathrm{Spec}(L) \to \mathrm{Spec}(k)\}, \mathbb{A}^1_k)$ and thus, by Theorem 5.2, it suffices to show that $\check{H}^1(\{\mathrm{Spec}(L) \to \mathrm{Spec}(k)\}, \mathrm{Aut}(\mathbb{A}^1_k))$ is a singleton.

To make this computation note first that we have the following short exact sequence of sheaves on $\mathrm{Spec}(k)_{\mathrm{Fl}}$:

$$1 \to \mathbf{G}_{a,k} \to \mathrm{Aut}(\mathbb{A}^1_k) \to \mathbf{G}_{m,k} \to 1 \tag{50}$$

essentially because the automorphisms of $\mathbb{A}^1_k$ are affine linear transformations. Thus, considering Theorem 3.35 we get an exact sequence of pointed sets

$$\check{H}^1(\mathcal{U}, \mathbf{G}_{a,k}) \to \check{H}^1(\mathcal{U}, \mathrm{Aut}(\mathbb{A}^1_k)) \to \check{H}^1(\mathcal{U}, \mathbf{G}_{m,k}) \tag{51}$$

where $\mathcal{U} = \{\mathrm{Spec}(L) \to \mathrm{Spec}(k)\}$. But, these outer cohomology groups are easy to compute! Namely, we know that $H^1(\mathrm{Spec}(k)_{\mathrm{Fl}}, \mathbf{G}_{a,k})$ classifies Zariski locally trivial $\mathbf{G}_{a,k}$ torsors on $\mathrm{Spec}(k)$ by Theorem 4.14 which *must* be trivial—the Zariski site of $\mathrm{Spec}(k)$ is trivial! Moreover, we know from Theorem 4.2 that $\check{H}^1(\mathcal{U}, \mathbf{G}_{m,k})$ must be trivial since it's a subgroup of $\check{H}^1(\mathrm{Spec}(k)_{\mathrm{Fl}}, \mathbf{G}_{m,k})$ which is classifying line bundles on $\mathrm{Spec}(k)$ of which there are obviously only the trivial one!

So, putting all of the components of the previous paragraph together, we do indeed see that if Y is geometrically $\mathbb{A}^1_k$ (i.e. $Y_{k^{\mathrm{sep}}} \cong \mathbb{A}^1_{k^{\mathrm{sep}}}$). Note that this also gives an indication about why the following somewhat surprising statment is true. As far as I know, the analagous question for affine n-space for $n > 2$ is *open* in most cases. This is somewhat not shocking since, as the above shows, this really comes down to computing cohomology of the automorphism sheaf of $\mathbb{A}^n_k$ and this is an incredibly mysterious object (cf. the Jacobian conjecture).

Central simple algebras and Brauer-Severi varieties Let's fix a field k. The group $\mathrm{PGL}_{n,k}$, defined as the fppf sheaf quotient of $\mathrm{GL}_{n,k}/\mathbf{G}_{m,k}$ (where $\mathbf{G}_{m,k}$ is embedded diagonally) is the automorphism sheaf of two a priori unrelated objects. Namely, $\mathrm{PGL}_{n,k} = \mathrm{Aut}(\mathrm{Mat}_n(k))$ (its the automorphism group of the non-commutative algebra of $n \times n$-matrices) and $\mathrm{PGL}_{n,k} = \mathrm{Aut}(\mathbb{P}^{n-1}_k)$ (its the automorphisms group of $(n-)1$-dimensional projective space).

Thus, we see that $\mathrm{PGL}_{n,k}$-torsors are twists of $\mathrm{Mat}_n(k)$, so-called *central simple k-algebras*, as well as twists of $\mathbb{P}^{n-1}_k$, so-called *Brauer-Severi varieties*. I would say more on this topic if there didn't exist an exceptionally good book discussing both of these objects and their explicit relationship. Namely, see the book *Central simple algebras and Galois cohomology* by Gille and Szamuely.

Algebraic groups Note that fibered category of algebraic groups over X_{Fl} is a stack—this follows since Aff is a stack, and the necessary commutativity of the diagrams defining the multiplication maps can be checked flat locally.

Thus, we have the following result:

Theorem 5.4: *Let X be a scheme and T an X-scheme. Then, if G_0 is an algebraic T-group, the pointed set* $\mathrm{Twist}(G_0)$ *of algebraic T-groups locally (in the flat topology on T) isomorphic to G_0 is isomorphic to the pointed set $\check{H}^1(T, \mathrm{Aut}(G_0))$. In particular, for any cover $\{U_i \to T\}$ the twists* $\mathrm{Twist}(\{U_i\}, G_0)$ *of G_0 which become isomorphic to G on each U_i is isomorphic, as a pointed set, to $\check{H}^1(\{U_i\}, \mathrm{Aut}(G_0))$.*

As an example one might try and consider the twists of $\mathrm{GL}_{n,k}$ over a field k. One then wants to understand $\check{H}^1(\mathrm{Spec}(k)_{\text{ét}}, \mathrm{Aut}(\mathrm{GL}_{n,k}))$. Well one can show that as a sheaf one has a canonical short exact sequence

$$1 \to \mathrm{PGL}_{n,k} \to \mathrm{Aut}(\mathrm{GL}_{n,k}) \to \underline{\mathbb{Z}/2\mathbb{Z}} \to 1 \tag{52}$$

which is non-canonically split. Thus we get an exact sequence of pointed sets

$$1 \to \check{H}^1(\mathrm{Spec}(k)_{\text{ét}}, \mathrm{Aut}(\mathrm{GL}_{n,k})) \to \mathsf{Twist}(\mathrm{GL}_{n,k}) \to H^1(\mathrm{Spec}(k)_{\text{ét}}, \mathbb{Z}/2\mathbb{Z}) \to 1 \tag{53}$$

We now from the previous section that $\check{H}^1(\mathrm{Spec}(k)_{\text{ét}}, \mathrm{PGL}_{n,k})$ is parameterizing central simple k-algebras Δ and we know from Corollary 4.12 that $H^1(\mathrm{Spec}(k)_{\text{ét}}, \underline{\mathbb{Z}/2\mathbb{Z}}) = \mathrm{Hom}_{\text{cont.}}(\mathrm{Gal}(k^{\text{sep}}/k), \mathbb{Z}/2\mathbb{Z})$ or, in other words, is parameterizing degree 2-extensions of k (as well as the trivial extension k/k).

What does this mean for $\mathsf{Twist}(\mathrm{GL}_{n,k})$? Well one can show that the twists of $\mathrm{GL}_{n,k}$ all look like $U(\Delta, *)$ where Δ is a central simple algebra over k or a degree 2-extension of k and $*$ is a so-called *involution of the second kind*. The group $U(\Delta, *)$ is the *unitary group* associated to such a setup. One then sees why an exact sequence like (53) makes sense.

For more information on this see the relevant part of Platonov and Rapinchuk's book *Algebraic groups and number theory* (i.e. §2.2 of Chapter 2).